Chapter 1. Basic Electrostatics: Charges in Vacuum

Much of the material covered in this chapter should be known to the students from their undergraduate studies. Because of that, the explanations are very brief. For the remedial reading, virtually any undergraduate E&M textbook may be used; I can recommend, e.g., D. J. Griffiths, *Introduction to Electrodynamics*, 3rd ed., Prentice-Hall, 1999.

1.1. Coulomb law and the superposition principle

A serious discussion of the Coulomb law (formulated in the 1780s by Charles Augustin de Coulomb) requires a common agreement on the meaning of the following notions:

- *electric charges* q_i,[1] as revealed, most explicitly, by experimental observation of *electrostatic forces* acting between the charged bodies whose relative speed is much less than the speed of light;[2]

- *space and time*, including the notion of coordinate systems ("reference frames"),

- mechanical *forces* which may be defined, for example, via the 2nd Newton law for particle motion in *inertial* coordinate systems – see, e.g., Chapter 1 of the Classical Mechanics part of my lecture notes; and

- a *point* electric charge, meaning a charged object so small that its size and shape are insignificant for the problem under study. The point's position is completely described (in a given reference frame) by its *radius-vector* $\vec{r} = \{x, y, z\}$.

I will assume that these notions are known to the reader (though my strong advice is to give some thought to their vital importance for this course). This allows us to formulate the (experimental!) *Coulomb law* for the electrostatic interaction of two point charges in vacuum:

$$\vec{F}_{12} = k\frac{q_1 q_2}{\left|\vec{r}_1 - \vec{r}_2\right|^3}(\vec{r}_1 - \vec{r}_2)\ . \qquad (1.1)$$

(Here index 12 denotes the force exerted on charge 1 by charge 2.) Several remarks are due here:

(i) Equation (1)[3] implies that a *point charge does not interact with itself*. This fact may look trivial, but becomes less evident (though still true) in quantum mechanics where the electric charge of even an elementary particle is effectively spread around some volume, together with particle's wavefunction. Moreover, there are some widely used approximations, e.g., the Kohn-Sham equations in

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1 One of the most important experimental facts is the *electric charge conservation*, mathematically meaning that the algebraic sum of all q_i inside any closed volume is conserved, unless the charged particles cross the volume's border. We will use (or imply) this law in many parts of this course.

2 In the modern metrology, the speed of light in vacuum is considered as a fixed (exact) number, $c = 2.99792458\times10^8$ m/s, consistent with the legacy standards of length and time, but postulating an exact relation for the re-definition of the meter and the second in more fundamental terms.

3 As in all other parts of my lecture notes, the chapter number is omitted in the references to equations, figures, and sections within the same chapter.

the density functional theory of condensed matter, which essentially violate this law (thus limiting their accuracy and applicability.

(ii) According to Eq. (1), the magnitude of the force, F_{12}, is inversely proportional to the square of the distance between the two charges – the well-known undergraduate-level formulation of the law.

(iii) Direction-wise, Eq. (1) gives the force exerted by charge 1 upon charge 2; according to the 3rd Newton law, the reciprocal force is equal in magnitude but opposite in direction: $\vec{F}_{21} = -\vec{F}_{12}$.

(iv) The constant k in Eq. (1) depends on the system of units. In the *Gaussian* ("CGS") units, most widely used in theoretical physics, k is set to one, for the price of introducing a special unit of charge (the "*statcoulomb*"). On the other hand, in the *international system* ("SI") of units, the charge unit is one *coulomb* (abbreviated C),[4] and k is different from unity:[5]

$$k = \frac{1}{4\pi\varepsilon_0} \equiv 10^{-7} c^2 . \tag{1.2}$$

I have to (regretfully) notice that the struggle between the proponents of these systems bears all the unfortunate signs of a religious war, with slim chances for any side to give up in any foreseeable future. In my humble view, both systems have their advantages and handicaps, and that *any educated physicist should know them both*. Following the recent (3rd) edition of the common textbook by J. D. Jackson, I will mostly use SI units in these notes, but will duplicate the key formulas in the Gaussian units.

Another experimental fact vital for electrostatics (and independent of the Coulomb law as formulated above) is the *linear superposition principle*: the electrostatic forces exerted on some point charge (say, q) by other charges do not affect each other and add up as vectors to form the net force:

$$\vec{F} = \sum_j \vec{F}_j , \tag{1.3}$$

where the summation is extended over all charges but q. Combining this equation with Eq. (1), we get

$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \sum_j \frac{q q_j}{\left|\vec{r} - \vec{r}_j\right|^3} \left(\vec{r} - \vec{r}_j\right) = q \frac{1}{4\pi\varepsilon_0} \sum_j \frac{q_j}{\left|\vec{r} - \vec{r}_j\right|^3} \left(\vec{r} - \vec{r}_j\right), \tag{1.4}$$

where $\vec{r}$ is the position of charge q. The second form of this equation implies that it makes sense to intrude the notion of the *electric field*

$$\vec{E}(\vec{r}) \equiv \frac{1}{4\pi\varepsilon_0} \sum_j \frac{q_j}{\left|\vec{r} - \vec{r}_j\right|^3} \left(\vec{r} - \vec{r}_j\right) \tag{1.5}$$

as some entity independent of the "probe" charge q, so that the force acting on the charge might be presented in a very simple form

$$\vec{F} = q\vec{E}. \tag{1.6}$$

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[4] In the formal metrology, one coulomb is defined as the charge carried over by constant current of one ampere (see Ch. 5 for its definition) during one second.

[5] Constant ε_0 is sometimes called the "*free space permittivity*"; from Eq. (2), $\varepsilon_0 \approx 8.854 \times 10^{-12}$ SI units.

This form is so appealing that Eq. (6) is used well beyond the boundaries of the electrostatics in free space. Moreover, the notion of field becomes virtually unavoidable in time-dependent phenomena (such as EM waves), where the electromagnetic field shows up as a specific form of matter, with zero rest mass, and hence different from the usual material particles.

Many problems involve so many closely located point charges that it is possible to approximate them with a continuous charge distribution with *volume* (3D) *density* ρ defined as[6]

$$\rho(r')d^3r' = dQ = \sum_{j\in d^3r'} q_j \,. \tag{1.7}$$

In this approximation, sum (5) turns into an integral:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{\left|\vec{r}-\vec{r}'\right|^3}(\vec{r}-\vec{r}')d^3r'. \tag{1.8}$$

This equation may be used even in the case of discrete point charges, if we use the notion of Dirac's δ-function which is a mathematical approximation for a very sharp function equal to zero everywhere but one point, and still having a finite (unit) integral. This function may be formally defined by equation

$$\int_V f(\vec{r}')\delta(\vec{r}'-\vec{r}_j)d^3r' = \begin{cases} f(\vec{r}_j), & \text{if } \vec{r}_j \in V, \\ 0, & \text{otherwise,} \end{cases} \tag{1.9}$$

where f is a smooth but otherwise arbitrary function of coordinates. Indeed, in this formalism, a set of point charges q_j located in points $\vec{r}_j$ may be presented by the pseudo-continuous distribution with density

$$\rho(\vec{r}) = \sum_j q_j \delta(\vec{r}-\vec{r}_j). \tag{1.10}$$

Plugging this expression into Eq. (8) and using the definition (9), we come back to the discrete version (5) of the Coulomb law.

1.2. The Gauss law and the first Maxwell equations

Due to this extension to discrete charges, it may seem that Eqs. (6) and (8) is all we need for solving any problem of electrostatics. This is not so, due to many reasons. Most superficially, the direct use of Eq. (8) frequently leads to complex calculations. Indeed, let us consider a very simple example: the electric field produced by a spherically-symmetric charge distribution with density $\rho(r')$. We may immediately use the problem symmetry to argue that the electric field should be also spherically-symmetric, with only one component in spherical coordinates: $\vec{E} = E(r)\vec{n}_r$, where $\vec{n}_r \equiv \vec{r}/r$ is the unit vector in the direction of the field observation point $\vec{r}$.

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[6] The 2D (areal) charge density σ and 1D (linear) density λ may be defined absolutely similarly: $dQ = \sigma d^2r$, $dQ = \lambda dr$. Note that a finite value of σ and λ means that the volume density ρ is infinite in the charge location points; for example for a plane z = 0, charged with a constant areal density σ, $\rho = \sigma\delta(z)$.

Taking this direction as the polar axis of a spherical coordinate system, we can use the evident independence of the elementary radial field dE (Fig. 1), created by the elementary charge $\rho(r')d^3r' = \rho(r')r'^2\sin\theta\, dr'\, d\theta' d\varphi'$, on the azimuth angle φ', and reduce integral (8) to

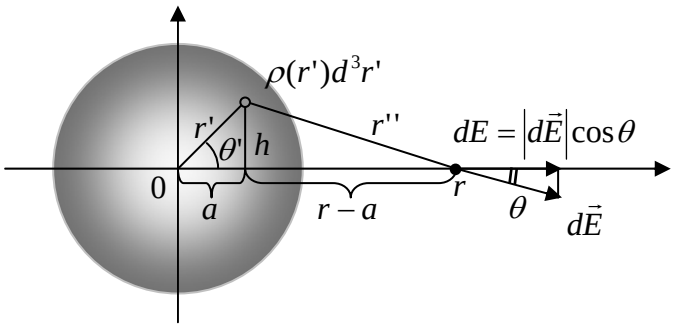

Fig. 1.1. One of the simplest problems of electrostatics: electric field produced by a spherically-symmetric charge distribution.

$$E = \frac{1}{4\pi\varepsilon_0} 2\pi \int_0^{\pi} \sin\theta' d\theta' \int_0^{\infty} r'^2 dr' \frac{\rho(r')}{(r'')^2}\cos\theta, \qquad (1.11)$$

where a, h, θ, and r'' are geometrical parameters defined in Fig. 1. Since they all may be expressed via r' and θ',

$$a = r'\cos\theta', \quad h = r'\sin\theta', \quad \theta = \arctan\frac{h}{r-a}, \quad (r'')^2 = h^2 + (r - r'\cos\theta)^2, \qquad (1.12)$$

integral (11) may be eventually reduced to a one-dimensional integral over r' (see below) but this would require some effort. For more complex problem, integral (8) may be much more complex, defying an analytical solution.

One could argue that with the present-day abundance of computers and numerical algorithm libraries, one can always resort to numerical integration. This argument may be enhanced by the fact that numerical *integration* is based on the replacement of the integral by a sum, and summation is much more robust to (unavoidable) discretization and rounding errors than the finite-difference schemes typical for the numerical solution of *differential* equations.

These arguments are only partly justified, since in many cases the numerical approach runs into a problem sometimes called the "*curse of dimensionality*", in which the last word refers to the number of input parameters of the problem to be solved. Let us discuss this issue (which is common for most fields of physics and, more generally, any quantitative science).

If the number of the parameters of a problem is small, the results of its numerical solution may be of the same (and in some sense higher) value than the analytical ones. For example, if a problem has no parameters, and its result is just one number (say, $\pi^2/4$), this "analytical" answer hardly carries more information than its numerical form 2.4674011… Now, if the problem has one input parameter (say, a), the result of an analytical approach in most cases may be presented as an analytical function $f(a)$. If the function is simple (say, $f(a) = \sin a$), this function gives us everything we want to know. However, if the function is complicated, you would need to calculate it numerically for a set of values of parameter a and possibly present the result as a plot. The same results (and the same plot) can be calculated numerically, without using analytics at all. This plot may certainly be very valuable, but since the analytical form has a potential of giving you more information (say, the values of $f(a)$ outside the plot

range, or the asymptotic behavior of the function), it is hard to say that the numerics completely beat the analytics here.

Now let us assume that you have more input parameters. For two parameters (say, *a* and *b*), instead of one curve *f*(*a*) you would need a family of such curves for several (sometimes many) values of *b*. Still, the plots sometimes may fit one page, so it is still not too bad. Now, if you have three parameters, the full representation of the results may require many pages (maybe a book) full of curves, for four parameters we may speak about a bookshelf, for five parameters something like a library, etc. For large number of parameters, typical for many scientific problems, the number of points in the parameters space grows exponentially, even the volume of calculations necessary for the generation of this data may become impracticable, despite the dirt-cheap CPU time we have now.

Thus, despite the current proliferation of numerical methods in physics, and with all due respect to them,[7] I believe that analytical results have an ever-lasting value, and we should try to get them whenever we can. For our current problem of finding electric field generated by a fixed set of electric charges, large help comes from the *Gauss law*.

Let us consider a single point charge *q* inside a smooth, closed surface *A* (Fig. 2), and calculate the product $E_n dA$ where *dA* is an infinitesimal element of the surface (which may be well approximated with a plane of that area), and E_n is the component of the electric field in that point, normal to that plane. This component can of course be calculated as $E \cos\theta$, where θ is the angle between vector $\vec{E}$ and the unit vector $\vec{n}$ normal to the surface. (Alternatively and equivalently, E_n may be presented as the scalar product $\vec{E} \cdot \vec{n}$.) Now let us notice that the product $\cos\theta\, dA$ is nothing more than the area *dA'* of a projection of *dA* onto the direction of vector $\vec{r}$ connecting charge *q* with this point of the surface (Fig. 2), because the angle between the planes *dA'* and *dA* is also equal to θ.

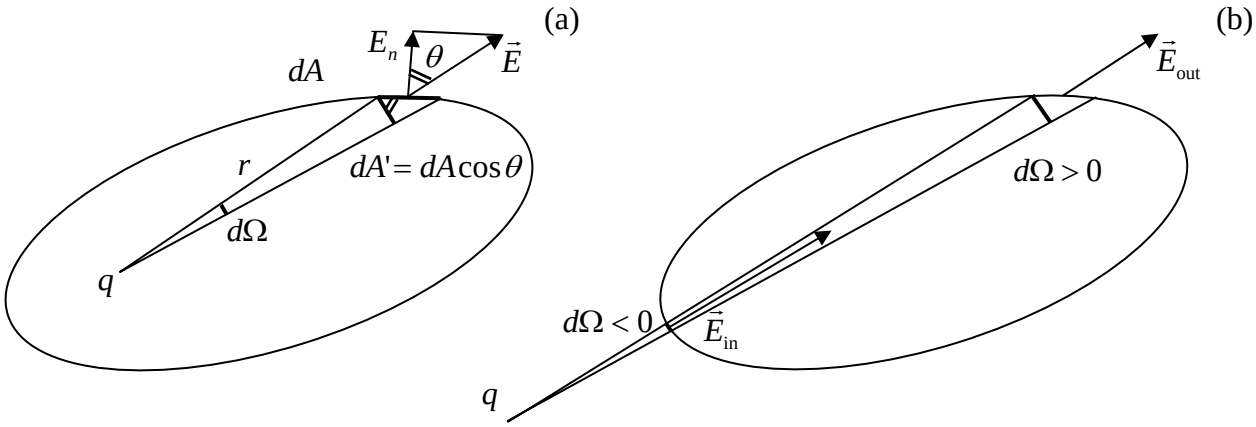

Fig. 1.2. To the derivation of the Gauss law: (a) the point charge is inside volume *V* and (b) outside of that volume.

Using for *E* the Coulomb law, we get

[7] Later in this chapter, I will give a brief introduction to the finite-difference methods applicable for the solution of boundary problems of electrodynamics.

$$E_n dA = E\cos\theta dA = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dA'. \qquad (1.13)$$

But the ratio dA'/r^2 is nothing more than the solid angle $d\Omega$ under which areas dA' and dA are seen from the charge point, so that $E_n dA$ may be presented as just a product of $d\Omega$ and a constant ($q/4\pi\varepsilon_0$). Integrating it over the whole surface, we get

$$\oint_A E_n d^2 r = \frac{q}{4\pi\varepsilon_0}\oint d\Omega = \frac{q}{\varepsilon_0}, \qquad (1.14)$$

since the full solid angle equals 4π. (The integral in the LHP of this equation is called the (full) *flux* of electric field through surface A.) Equation (14) expresses the Gauss law for one point charge. However, it is only valid if the charge is located inside the volume limited by the surface. In order to find the flux created by a charge outside of the surface, we still can use all the equations leading to Eq. (13), including that equality, but to proceed we have to be careful with the signs of the elementary contributions $E_n dA$. Namely, the unit vector $\vec{n}$ should always point out of the volume we consider (the so-called outer normal), so that the elementary product $E_n dA = (\vec{E}\cdot\vec{n})dA$ and hence $d\Omega = E_n dA/r^2$ are positive if vector $\vec{E}$ is pointing out of the volume (like in the example shown in Fig. 2a and the upper area in Fig. 2b), and negative in the opposite case (for example, the lower area in Fig. 2b). As the latter figure shows, if the charge is located outside of the volume, for each positive contribution $d\Omega$ there is always equal and opposite contribution to the integral. As a result, at the integration over the solid angle the positive and negative contributions cancel exactly, and

$$\oint E_n dA = 0. \qquad (1.15)$$

In order to reveal the real power of the Gauss law, let us generalize it to the case of many charges within volume V. Since the calculation of flux is a linear operation, the linear superposition principle (3) means that the flux created by several charges equals the (algebraic) sum of individual fluxes from each charge, for which either Eq. (14) or Eq. (15) are valid, depending on the charge position (in or out of the volume). As the result, for the total flux we get:

$$\oint_A E_n d^2 r = \frac{Q_V}{\varepsilon_0} \equiv \frac{1}{\varepsilon_0}\sum_{j\in V} q_j = \frac{1}{\varepsilon_0}\int_V \rho(\vec{r}')d^3 r', \qquad (1.16)$$

where Q_V is the net charge inside volume V. This is the full Gauss law. In order to understand its problem-solving power, let us return to the problem presented in Fig. 1, and apply Eq. (16) to a sphere of radius r, with the center coinciding with the center of the charge distribution. Due to its symmetry, which had already been discussed above, the electric field is perpendicular to the surface of the sphere in each point (i.e., $E_n = E$), and its magnitude is the same at all points: $E_n = E = E(r)$. As a result, the flux calculation is trivial:

$$\oint E_n d^2 r = 4\pi r^2 E(r), \qquad (1.17)$$

Now, applying the Gauss law (16), we get:

$$4\pi r^2 E(r) = \frac{1}{\varepsilon_0}\int_{r'<r} \rho(r')d^3 r' = \frac{4\pi}{\varepsilon_0}\int_0^r r'^2\,\rho(r')dr', \qquad (1.18)$$

so that, finally,

$$E(r) = \frac{1}{r^2 \varepsilon_0} \int_0^r r'^2 \, \rho(r') dr' = \frac{1}{4\pi\varepsilon_0} \frac{Q(r)}{r^2}, \tag{1.19}$$

where $Q(r)$ is the full charge inside the sphere of radius r:

$$Q(r) \equiv 4\pi \int_0^r \rho(r') r'^2 \, dr'. \tag{1.20}$$

In particular, this formula shows that the field outside of a sphere of a finite radius R is exactly the same as if all its charge $Q = Q(R)$ was concentrated in the sphere's center. (Note that this important result is only valid for the spherically-symmetric charge distribution.)

For the field inside the sphere, finding electric field still requires integration (20), but this 1D integral is much simpler that the 2D integral (11), and in some important cases may be taken analytically. For example, if charge Q is uniformly distributed inside a sphere of radius R,

$$\rho(r') = \text{const} = \frac{Q}{V} = \frac{Q}{(4\pi/3)R^3}, \tag{1.21}$$

Then the integration is elementary:

$$E(r) = \frac{1}{r^2 \varepsilon_0} \int_0^r r'^2 \, \rho(r') dr' = \frac{1}{4\pi\varepsilon_0} \frac{Qr}{R^3}. \tag{1.22}$$

We see that in this case the field is growing linearly from the center to the sphere's surface, and only then starts to decrease in agreement with the Coulomb law. Another important observation is that our results for $r < R$ and $r > R$ give the same value ($Q/4\pi\varepsilon_0 R^2$) at the charged sphere's surface, so that the electric field is continuous.

In order to understand this fact better, let us consider one more elementary example of the Gauss law's application. Consider a thin plane sheet (Fig. 3) charged uniformly, with areal density σ = const (see Footnote 5 above).

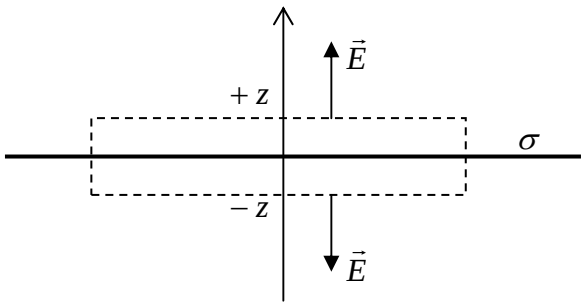

Fig. 1.3. Electric field of a charged plane.

In this case, it is fruitful to use the Gauss volume in the form of a planar "pillbox" of thickness 2z (where z is the Cartesian coordinate perpendicular to charged plane) – see Fig. 3. Due to the symmetry of the problem, it is evident that the electric field should be: (i) directed along axis z, (ii) constant on each of the upper and bottom side of the pillbox, (iii) equal and opposite on these sides, and (iv) vanish on the side surfaces of the box. As a result, the electric field flux is just $2AE(z)$, so that the Gauss law (16) yields

$$2AE(z) = \frac{1}{\varepsilon_0} Q(A) = \frac{1}{\varepsilon_0} \sigma A, \tag{1.23}$$

and we get a very simple (but very important) formula

$$E(z) = \frac{\sigma}{2\varepsilon_0} = \text{const}. \tag{1.24}$$

Note that, somewhat, counter-intuitively, the field magnitude does not depend on the distance from the charged plane. From the point of view of the Coulomb law (5), this result may be explained as follows, the farther the observation point from the plane, the weaker the effect of each elementary charge, $dQ = \sigma d^2 r$, but the more such elementary charges give contributions to the vertical component of vector $\vec{E}$.

Note also that though the magnitude $E \equiv \left|\vec{E}\right|$ of the electric field is constant, its vertical component E_z changes sign at $z = 0$ (Fig. 3), experiencing a "*discontinuity*" (jump) equal to $\Delta E_z = \sigma/\varepsilon_0$. This jump disappears if the surface is not charged ($\sigma = 0$). This statement remains true in a more general case of finite volume (but not surface!) charge density ρ. Returning for a minute to our charged sphere problem, very close to its surface it may be considered plane, so that the electric field should indeed be continuous, as it is.

In order to complete our discussion of the Gauss law, let us mention what is sometimes called its *differential form*. It may be obtained from the "*integral form*" (16) of this law, using the following *divergence theorem* which, according to the vector algebra, is valid for any continuous vector, in particular $\vec{E}$, and for any volume V limited by closed surface A:

$$\oint_A E_n d^2 r = \int_V (\vec{\nabla} \cdot \vec{E}) d^3 r \,. \tag{1.25}$$

(The scalar product in the RHP of this equation is nothing more that the divergence of vector $\vec{E}$, i.e. just the sum of its first partial derivatives over all three Cartesian coordinates.) Combining Eq. (25) with the Gauss law (16), we get[8]

$$\int_V \left[\vec{\nabla} \cdot \vec{E} - \frac{\rho}{\varepsilon_0} \right] d^3 r = 0. \tag{1.26}$$

For a given distribution of electric charge (and hence of the electric field), this equation should be valid for any choice of volume V. This can only be true if the function under the integral vanishes at each point, i.e.[9]

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}. \tag{1.27}$$

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[8] Note that in a sharp contrast with the integral form (16), Eq. (26) is *local*: it relates the electric field divergence to the charge density *in the same point*.

[9] Due to the key importance of this relation, it is useful to remember it in the Gaussian units as well: $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$.

This equation is sometimes called the differential form of the Gauss law, but it may be also thought about as the differential form of the Coulomb law (8). I believe that a much better choice is to call it the (stationary version of) the first Maxwell equation. One more of such embryos may be obtain by noticing that for the Coulomb field of a single point charge (and hence, due to the linear superposition principle, for the electrostatic field of any charge system),

$$\vec{\nabla} \times \vec{E} = 0. \tag{1.28}$$

(We will arrive at two other Maxwell equations in Chapter 5, and then generalize them to their full, time-dependent form.)

Just to feel Eq. (27) better, let us apply it to the same example of a uniformly charged sphere (Fig. 1). Vector algebra teaches us in the case of spherical symmetry ($\vec{E} = E(r)\vec{n}_r$) the divergence of any differentiable vector field may be simply expressed in spherical coordinates:

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{r^2}(r^2 E)', \tag{1.29}$$

where the prime sign denotes the differentiation over r. As a result, Eq. (27) yields a linear, ordinary differential equation for the function $E(r)$:

$$\frac{1}{r^2}(r^2 E)' = \begin{cases} \rho/\varepsilon_0, & \text{for } r \le R, \\ 0, & \text{for } r \ge R, \end{cases} \tag{1.30a}$$

which may be readily integrated on each of the segments:

$$E(r) = \frac{1}{\varepsilon_0}\frac{1}{r^2} \times \begin{cases} \rho \int r^2 dr = \rho r^3/3 + C_1, & \text{for } r \le R, \\ C_2, & \text{for } r \ge R. \end{cases} \tag{1.30b}$$

In order to determine the integration constant C_1, we should use boundary condition $E(0) = 0$. (It follows from the symmetry: in the center of the sphere, electric field has to vanish, because otherwise which side would it be directed?) Constant C_2 may be found from the continuity condition $E(R\text{-}0) = E(R\text{+}0)$, which has already been discussed above. As a result, we arrive at to our old results (19) and (22).

We can see that in this particular case, using the differential form of the Gauss law is more complex than its integral form. (For our second example, shown in Fig. 3, it is even less natural – see Exercise 1.) However, Eq. (27) and its generalizations may be invaluable in the cases where the charge distribution in not known a priori and has to be found in a self-consistent way. (We will discuss many such cases below.)

1.3. Scalar potential and energy of the electric field

One more help for solving electrostatics (and more complex) problems may be obtained from the notion of the *electrostatic potential* which is just the potential energy U of a charged particle, normalized on its charge:[10]

[10] This definition is kept in any system of units.

$$\Phi \equiv \frac{U}{q}. \tag{1.31}$$

(In electro*statics*, both U and Φ are functions of the charge position $\vec{r}$, but not time.)

The reason for the introduction of these notions is given by classical mechanics (see, e.g., Sec. 1.4 in the CM part of my notes). Briefly, the use of potential energy allows us to solve many problems much more easily. (Later in this section, just one but very convincing example will be given.) The benefits of this formalism are especially large for the case of *potential forces*, for example those depending just on the particle position. Equations (6) and (8) show that, in the static situations, the electric field clearly falls into this category.

For such field, the potential energy may be defined as a scalar function $U(\vec{r})$ which allows the force to be calculated as

$$\vec{F} = -\vec{\nabla}U\,. \tag{1.32}$$

Dividing by the charge of the particle upon which the force is exerted, and using Eq. (31), we get[11]

$$\vec{E} = -\vec{\nabla}\Phi\,. \tag{1.33}$$

In order to find the scalar potential, let us start with the simplest case of a single point charge q placed at origin. For it, the Coulomb law (5) takes a simple form

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} q \frac{\vec{r}}{r^3} = \frac{1}{4\pi\varepsilon_0} q \frac{\vec{n}_r}{r^2}. \tag{1.34}$$

It is easy to check that the last fraction in the RHP of this equation is equal to $-\vec{\nabla}(1/r)$. (This may be either by Cartesian components or just using the well-known expression $\vec{\nabla}f = (df/dr)\vec{n}_r$ valid for any spherically-symmetric scalar function f.) Hence, according to the definition (33), for this particular case[12]

$$\Phi = \frac{1}{4\pi\varepsilon_0} \frac{q}{r}. \tag{1.35}$$

Note that we could add an arbitrary constant to this potential (and indeed to any other formula for Φ given below) without changing the force, but it is convenient to define the potential energy to be zero at infinity.

For a single charge in an arbitrary position (say, $\vec{r}_j$), r should be evidently replaced for $\left|\vec{r} - \vec{r}_j\right|$. Now, the linear superposition principle (3) allows for an easy generalization of this formula to the case of an arbitrary set of discrete charges,

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_j \frac{q_j}{\left|\vec{r} - \vec{r}_j\right|}. \tag{1.36}$$

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11 This relation is closely related to Eq. (28), because according to vector algebra, any gradient field has vanishing curl.

12 This fundamental formula looks even simpler in the Gaussian units: $\Phi = q/r$.

Finally, using the same arguments as in Sec. 1, we can use this result to argue that in the case of an arbitrary continuous charge distribution

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} d^3 r' \ . \qquad (1.37)$$

Again, the notion of the Dirac delta-function allows to use the last equation for discrete charges as well, so that Eq. (37) may be considered as the final result for the electrostatic potential.

Before going further, let us demonstrate how useful the notion of Φ is, on a very simple example. Let two similar charges q be launched from afar, with an initial velocity $v_0 << c$ each, straight toward each other (with zero impact parameter). Since, according to the Coulomb law, the charges repel, they will stop at some minimum distance $r_{\min}$ from each other, and will than fly back (Fig. 4).

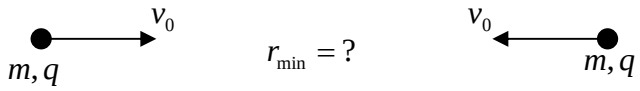

Fig. 1.4. A simple problem of electric particle motion.

We could of course find $r_{\min}$ directly from the Coulomb law. However, for that we would need to write the 2nd Newton law for each particle (actually, due to the problem symmetry they would be similar), then integrate them over time to find the particle velocity v as a function of distance, and then recover $r_{\min}$ from the requirement $v = 0$.

The notion of potential allows this problem to be solved in one line. Indeed, a particle's mechanical energy $E = T + U$ is conserved in the field of potential forces, and in our non-relativistic case, the kinetic energy T is just $mv^2/2$ and the potential energy $U = q\Phi$. Hence, equating the total energy of two particles in the points $r = \infty$ and $r = r_{\min}$, and using Eq. (35) for Φ, we get

$$2\frac{mv_0^2}{2} + 0 = 0 + \frac{1}{4\pi\varepsilon_0}\frac{q^2}{r_{\min}}, \qquad (1.38)$$

immediately giving us $r_{\min}$. One, however, may question whether we were right to count the potential energy only once, despite having two particles in the problem. The answer is yes, as may be explained by the following arguments.

The definitions (32), (33) may be of course re-written in the integral form:

$$U(\vec{r}) = -\int^{r} \vec{F}(\vec{r})\cdot d\vec{r} + \text{const}, \qquad \text{i.e. } \Phi(\vec{r}) = -\int^{r} \vec{E}(\vec{r})\cdot d\vec{r} + \text{const}, \qquad (1.39)$$

Let us use the fact that the potential energy does not depend on the way the configuration has been created, and consider the following process. First, let us move one charged particle (say, q_1) from infinity to an arbitrary point of space ($\vec{r}_1$) in the absence of other charges. Equations (39) show that since during the motion the particle does not experience any force (remember, the charge does not interact with itself!), its potential energy is the same as at infinity (with our choice of the arbitrary constant, zero): $U_1 = 0$. Now let us fix the position of that charge, and move in another charge (q_2) from infinity to

point $\vec{r}_2$. This particle does experience the force excerted by q_1 during its motion, so that its contribution to the final potential energy

$$U_2 = q_2 \Phi_1(\vec{r}_2). \tag{1.40}$$

Since the first particle was not moving during this process, the total potential energy U of the system is equal to just U_2. This is exactly the fact which we used when writing Eq. (38). (Prescribing a similar energy to charge q_1 as well would constitute a very popular error called the *double-counting*.)

Let us extend these arguments to calculate a very important expression for the potential energy of an arbitrary system of electric charges (Fig. 5). Fixing the first two charges in points $\vec{r}_1$ and $\vec{r}_2$, respectively, and bringing in the third charge (slowly, in order to avoid any magnetic field effects!) from infinity, we increment the potential energy by

$$U_3 = q_3 \left[\Phi_1(\vec{r}_3) + \Phi_2(\vec{r}_3) \right]. \tag{1.41}$$

It is clear how to generalize this result to an arbitrary (jth) charge:

$$U_j = q_j \left[\Phi_1(\vec{r}_j) + \Phi_2(\vec{r}_j) + \Phi_3(\vec{r}_j) + ... \right] = q_j \sum_{k<j} \Phi_k(\vec{r}_j) \tag{1.42}$$

(Notice the condition $k < j$ which suppresses the erroneous double-counting.)

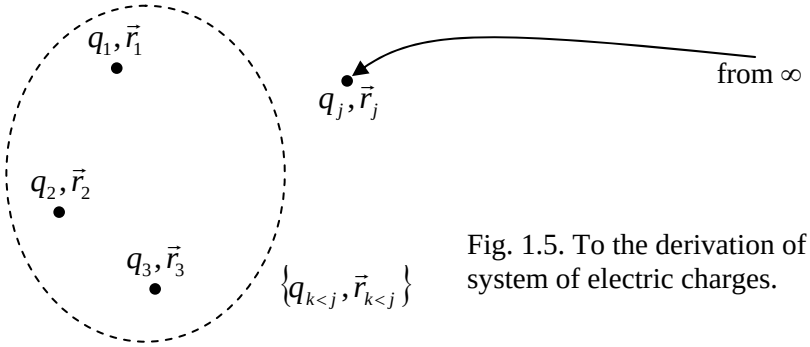

Fig. 1.5. To the derivation of the potential energy of a system of electric charges.

Now, summing up all the increments, for the total electrostatic energy of the system we get:

$$U = \sum_j U_j = \sum_{\substack{j,k \\ k<j}} q_j \Phi_k(\vec{r}_j). \tag{1.43}$$

This is a very important formula, but for its generalization to the continuous charge case it is better to rewrite it in a more symmetric form using Eq. (35):

$$U = \frac{1}{4\pi\varepsilon_0} \sum_{\substack{j,k \\ k<j}} \frac{q_k q_j}{\left| \vec{r}_j - \vec{r}_k \right|}. \tag{1.44}$$

The expression under the sum is evidently symmetric with respect to the index swap, so that (ignoring the way Eq. (44) has been derived), we can rewrite it in a fully symmetric form

$$U = \frac{1}{8\pi\varepsilon_0} \sum_{\substack{j,k \\ k\neq j}} \frac{q_k q_j}{\left| \vec{r}_j - \vec{r}_k \right|} \tag{1.45}$$

which is now easily generalized to the continuous case:

$$U = \frac{1}{8\pi\varepsilon_0}\int d^3r \int d^3r' \frac{\rho(\vec{r})\rho(\vec{r}')}{\left|\vec{r}-\vec{r}'\right|}. \tag{1.46}$$

Note that in this case the restriction expressed in the discrete charge case as $k \neq j$ is not important, because if the charge density is a continuous function, integral (46) does not diverge at point $\vec{r} = \vec{r}'$.[13]

To some extent, Eq. (46) is the most explicit result for the electrostatic field energy, but sometimes it is useful to rewrite it in a different form. For that, let us notice that according to Eq. (37), the integral over *r'* in Eq. (46) (divided by $4\pi\varepsilon_0$) is just the full electrostatic potential at point r, and hence

$$U = \frac{1}{2}\int d^3r \rho(\vec{r})\Phi(\vec{r}). \tag{1.47}$$

For the discrete charge case, this becomes

$$U = \frac{1}{2}\sum_j q_j \Phi(\vec{r}_j), \tag{1.48}$$

but now it is important to remember that the "full" potential's value $\Phi(\vec{r}_j)$ should exclude the (infinite) contribution of charge *j* itself.

Equations (46) and (47) seem to imply that the contribution into the electrostatic field energy come only from the regions where electric charge is located. However, one of the beautiful features of physics is that sometimes completely different views at the same problem are possible. In order to get an alternative view of our current problem, let us first recast Eq. (37) into a differential form of Eq. (37). It is easy to derive it by plugging the definition Eq. (33) into the differential form (27) of the Gauss law:

$$\vec{\nabla}(\vec{\nabla}\Phi) \equiv \nabla^2\Phi = -\frac{\rho}{\varepsilon_0}. \tag{1.49}$$

This it the famous *Poisson equation* – so convenient for applications that even its particular case for $\rho = 0$,

$$\nabla^2\Phi = 0 \tag{1.50}$$

has earned a special name – the *Laplace equation* (and the differential operation denoted as ∇^2, a special name of the *Laplace operator*). The Poisson equation may be less convenient than the Gauss law for the calculation of the field of a fixed charge distribution $\rho(\vec{r})$, but is indispensable for many problems in which the distribution should be found in a self-consistent way, simultaneously with the field – see the next chapter.

Now we can return to Eq. (47) for the potential energy, and plug into it the charge density expressed from the Poisson equation:

[13] Nevertheless, one should remember this restriction when returning from Eq. (46) back to the discrete form, using the Dirac delta-function.

$$U = -\frac{\varepsilon_0}{2}\int d^3 r \Phi \nabla^2 \Phi \,. \tag{1.51}$$

This expression may be integrated by parts:

$$U = -\frac{\varepsilon_0}{2}\left[\oint_A \Phi(\vec{\nabla}\Phi \cdot \vec{n})dA - \int_V d^3 r \left(\vec{\nabla}\Phi\right)^2\right], \tag{1.52}$$

where A is the closed surface limiting the integration volume V. If this surface is so far from the charges we consider that electric field on it is negligibly small (or if we integrate over all space), the surface integral vanishes, and using the basic relation (33) we get a very important formula

$$U = \frac{\varepsilon_0}{2}\int E^2 d^3 r \,. \tag{1.53}$$

It certainly invites an interpretation very much different than Eq. (47): it is natural to present it in the form[14]

$$U = \int u(\vec{r}) d^3 r, \qquad u(\vec{r}) = \frac{\varepsilon_0}{2} E^2(\vec{r}), \tag{1.54}$$

and treat $u(\vec{r})$ as the density of the potential energy of the electric field, which is continuously distributed over all the space where the field exists (rather than just where the charge sits). Of course, within the realm of *electrostatics*, Eqs. (47) and (54) are equivalent, but *electrodynamics* shows that the latter equation is more general, and that it is more adequate to associate energy with the field than with its sources (in our current case, electric charges).

[14] In the Gaussian units, $u = E^2/8\pi$.