ELECTENG 204LEC Engineering Electromagnetics
Lectures from 10th to 17th September 2001,
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What is **electromagnetics**?

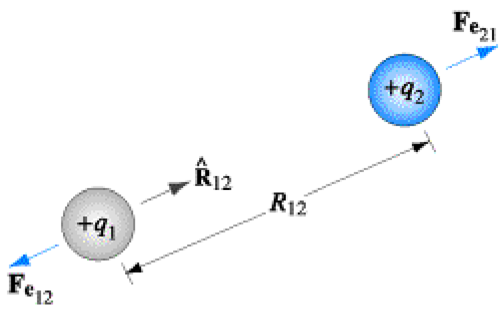
Electromagnetics is the study of the mutual interactions between **electric charges**. Charges may be stationary, they may move with constant velocity (v=const) or they may be in accelerated motion $\frac{dv}{dt} \neq 0$.

What is **electrostatics**?

Electrostatics deals with the interactions between stationary charges.

Key concepts and relationships are:

- The inverse square law **(Coulomb's law)**:

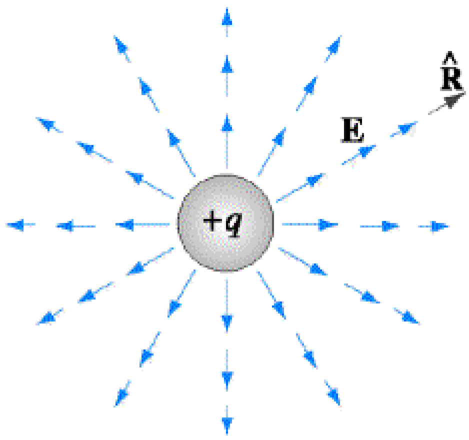


$$F_{e21} = \hat{R}_{12} \frac{q_1 q_2}{4\pi\varepsilon_0 R_{12}^2}$$

Where: F_{e21} is the electrical force acting on charge q_2 due to charge q_1, R_{12} is the distance between the two charges, $\hat{R}_{12}$ is a unit vector pointing from charge q_1 to charge q_2, and e_0 is a universal constant called the **electrical permittivity of free space** [e_0 = 8.854X10^{-12} F/m].

Read about the history of Coulomb's discovery at:
http://www.ele.auckland.ac.nz/~kacprzak/Coulomb/coulomb.htm

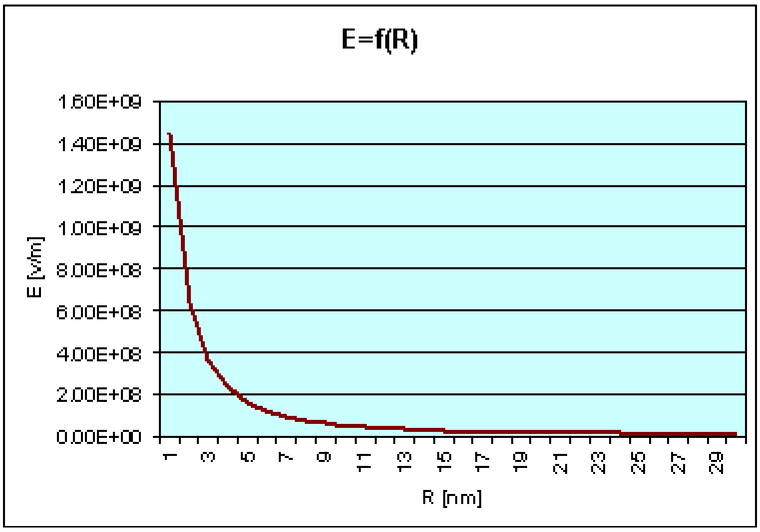
· Electric field and electric field strength (intensity) **E**



$$\mathbf{E} = \hat{\mathrm{R}} \frac{q}{4\pi\varepsilon_0 R^2} \qquad (V/m)$$

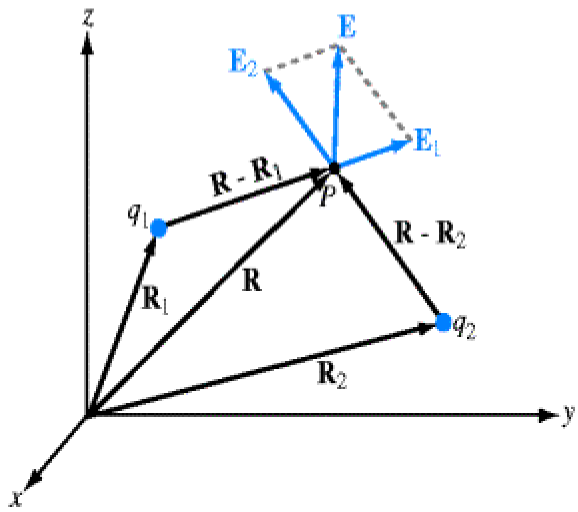
Where: R is the distance between the charge and the observation point, $\hat{\mathrm{R}}$ is the radial unit vector pointing away from the charge.

EXAMPLE: Electric field intensity from q=1.6X10^{-19} C (electron):



Example: Electric Field due to Two Point Charges

Two point charges q_1=2x10^{-5}C and q_2=2x10^{-5}C are located in free space at (1,3,-1) and (3,1,-2), respectively, in a Cartesian coordinate system.



Find (a) the electric field **E** at (3,1,-2) and (b) the force on a 8x10^{-5}C change located at that point, All distances are in meters.

Solution:

$$\mathrm{E} = \mathrm{E}_1 + \mathrm{E}_2 = \frac{1}{4\pi \cdot \varepsilon_0}\left[\frac{q_1(R-R_1)}{\left|R-R_1\right|^3} + \frac{q_2(R-R_2)}{\left|R-R_2\right|^3}\right]$$

The vectors R_1, R_2 and R are given by

$$R_1 = \hat{x}1 + \hat{y}3 - \hat{z}1$$
$$R_2 = -\hat{x}3 + \hat{y}1 - \hat{z}2$$
$$R = \hat{x}3 + \hat{y}1 - \hat{z}2$$

Hence,

$$\mathrm{E} = \frac{1}{4\pi \cdot \varepsilon_0}\left[\frac{2\times10^{-5}[\hat{x}(3-1)+\hat{y}(1-3)+\hat{z}(-2+1)]}{(\sqrt{(3-1)^2+(1-3)^2+(-2+1)^2})^3} + \frac{-4\times10^{-5}[\hat{x}(3+3)+\hat{y}(1-1)+\hat{z}(-2+2)]}{(\sqrt{(3+3)^2+(1-1)^2+(-2+2)^2})^3}\right]$$

Thus

$$\mathrm{E} = \frac{\hat{x}1 - \hat{y}4 - \hat{z}2}{108\pi \cdot \varepsilon_0} \times 10^{-5} \ \text{(V/m)}$$

(b)

$$\mathrm{F} = \mathrm{q}_3\mathrm{E} = 8\times10^{-5} \times \frac{\hat{x}1 - \hat{y}4 - \hat{z}2}{108\pi \cdot \varepsilon_0} \times 10^{-5} = \frac{\hat{x}2 - \hat{y}8 - \hat{z}4}{27\pi \cdot \varepsilon_0} \times 10^{-10} \quad \text{(N)}$$

· Electric potential **V**

In order to bring two charges near each other work must be done. In order to separate two opposite charges, work must be done.

$$V = \frac{W}{q_{moved}} \qquad \text{(V)}$$

Work or energy can be measured in Joules and charge is measured in Coulombs so the electrical potential can be measured in Joules per Coulomb, which has been defined as a volt.

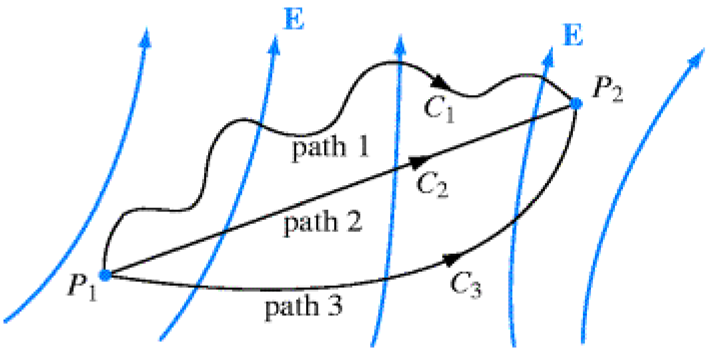
The differential electric potential energy dW per unit charge is called the differential electric potential (or differential voltage) dV. That is:

$$dV = \frac{dW}{q}$$

The potential difference between any two points P1, P2 is obtained by integrating

$$V_{21} = V_2 - V_1 = -\int_{P1}^{P2} E \cdot dl$$

along any path between them.
Where: dl is the vector differential distance



· Electric flux density **D**

In addition to the electric field intensity E, we will often find it convenient to also use a related quantity called the electric flux density D, given by

$$D = \varepsilon E \qquad (C/m^2)$$

In vacuum $D = \varepsilon_0 E$

· Gauss Law, which states that the integrated normal component of the electric flux crossing any closed surface is equal to the charge inside the surface

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = q_v$$

$$\nabla \cdot D = q_v$$

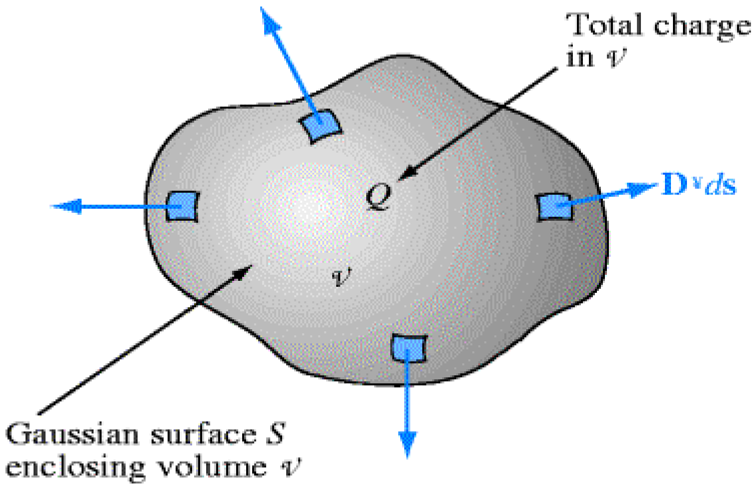
(Differential form of Gauss's law. The objective "differential" refers to the fact that the divergence operation involves spatial derivatives.)

It can be converted and expressed in integral form.

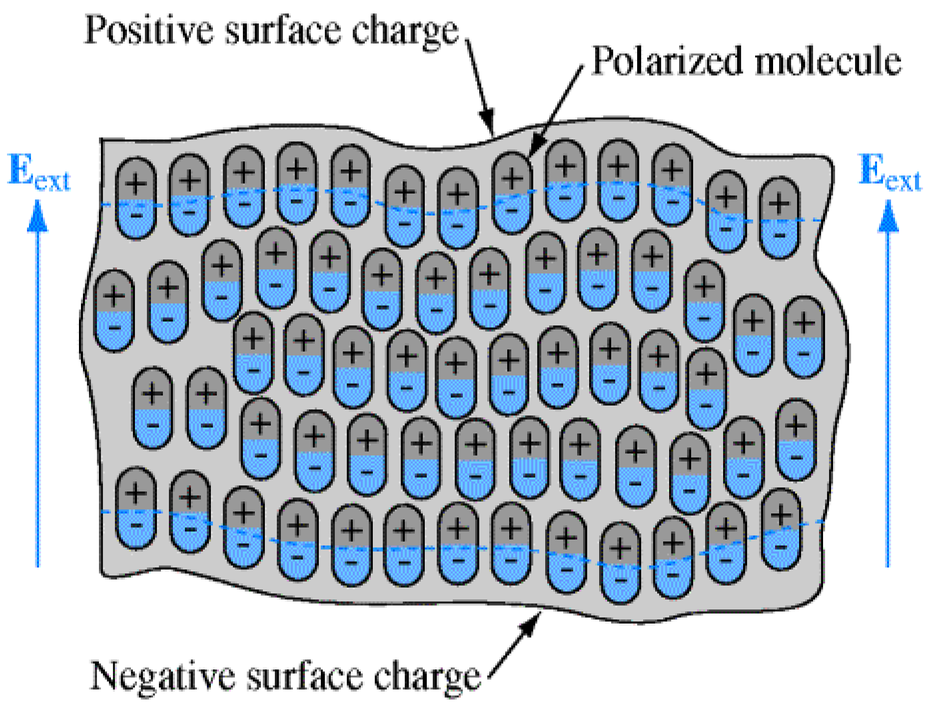
$$\int_v \nabla \cdot D dv = \int_v q_v dv = Q$$

which leads to

$$\oint_s D \cdot ds = Q$$



· The presence of a dielectric influence by the electric field due to the displacement of bound charges within the dielectric



· The energy density in an electric field is:

$$\frac{DE}{2} = \frac{\varepsilon E^2}{2} \qquad (\text{Joules/m}^3)$$

In a capacitor the energy may be expressed as $\frac{CV^2}{2}$ (Joules)

Magnetism

Magnetism developed independently of but in parallel with the science of electrostatics. It was based on the study of naturally occurring magnetic materials (principally lodestone). The basic experimental element was the **bar magnet** comprising 2 poles, which are in reality indivisible.

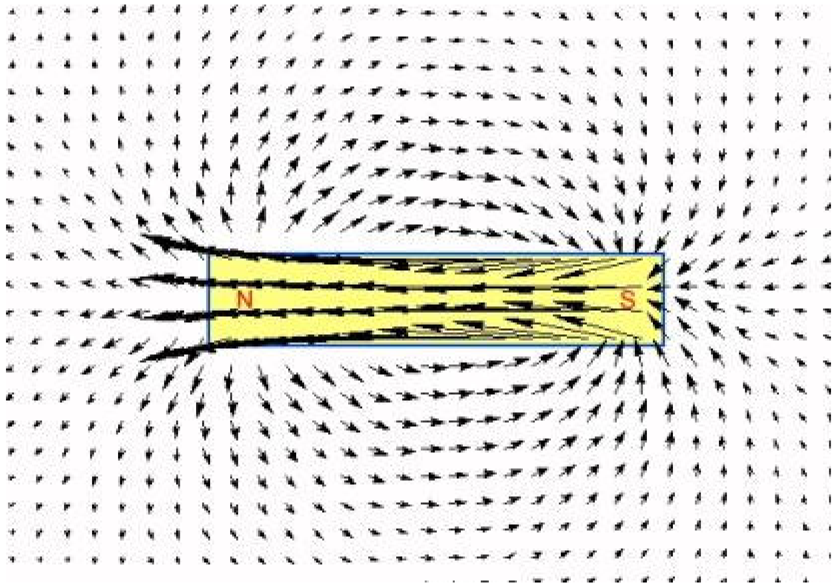


Figure Permanent magnet (from electromagnetic modeling)

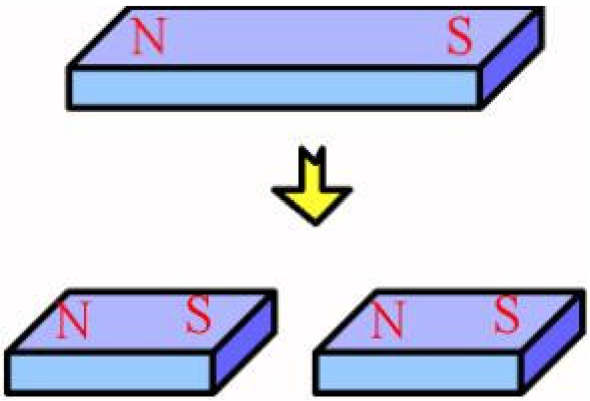


Figure Magnet before and after division

Electric charges can be isolated, but magnetic poles always exist in pairs and a small bar-magnet is effectively a magnetic dipole.

Concepts:

1. Inverse square law $F = \frac{Q_1 Q_2}{4\pi \cdot \mu_0 r^2}$, where Q1 and Q2 are the magnetic pole strengths and μ_0 is the magnetic permeability of free space [4px10^{-7} H/m].

2. The concept of a **Magnetic Field** followed, having strength **H** at a point where $F = HdQ, \quad H = \frac{Q}{4\pi \cdot \mu_0 r^2}$ and **H** in the **Magnetic Field Intensity.**

In a complementary way to electrostatics it was found convenient to define a new quantity **B** the **Magnetic Flux Density**, such that in vacuum:

$$\mathbf{B} = \mathrm{m}_0\mathbf{H}$$

We might note that in the presence of magnetic materials the relationship between **B** and **H** may be written

$$\mathbf{B} = \mathrm{m}\mathbf{H} = \mathrm{m}_0\mathrm{m}_r\mathbf{H}$$

Where: m_r Is the relative permeability of the material.

And also magnetic flux linking the surface S is defined as the total magnetic flux density passing through S, or

$$\Phi = \int_S B \cdot ds$$

and from Stoke's theorem,

$$\Phi = \oint_C A \cdot dl$$

where C is the contour boarding the surface S.

Over a close surface S Gauss's Law provides (since the total pole strength is always zero)

$$\nabla \cdot B = 0 \Leftrightarrow \int_s B \cdot ds = 0$$

A magnetic field not only exists around permanent magnets but can also be created by electric circuit.

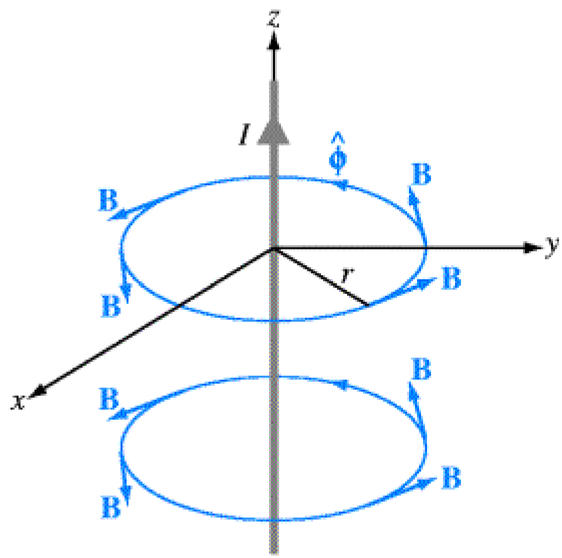


Figure The magnetic field induced by a steady current flowing in the z-direction

Magnetic flux density B near the conductor with a current (Biot-Savart law)

$$B = \hat{\varphi}\frac{\mu_0 I}{2\pi \cdot r} \qquad \text{(T)}$$

where: r is the radial distance from the current and $\hat{\varphi}$ is an azimuthal unit vector, μ_0 is the magnetic permeability of free space [4px10^{-7} H/m].

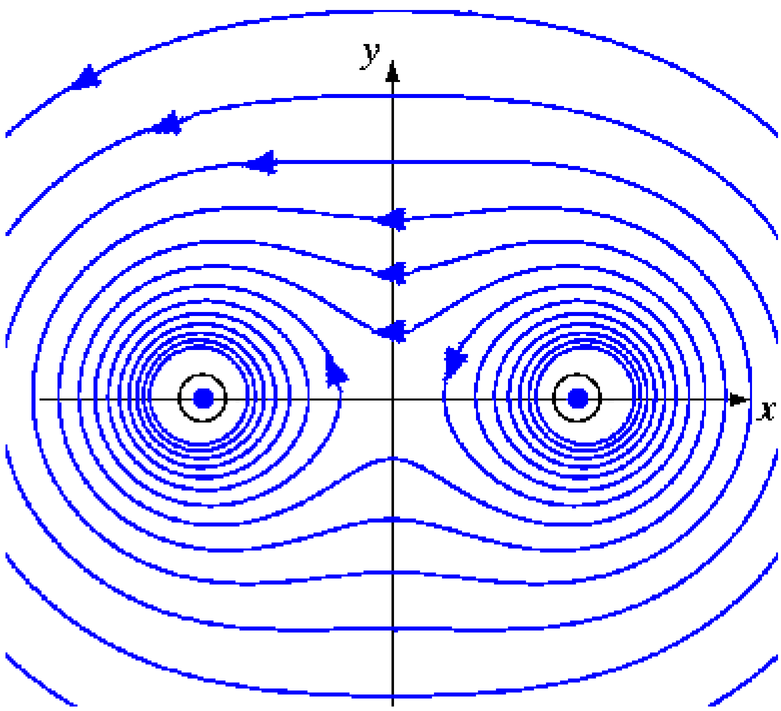


Figure Lines representing magnetic field - two parallel conductors carrying current in Z-direction

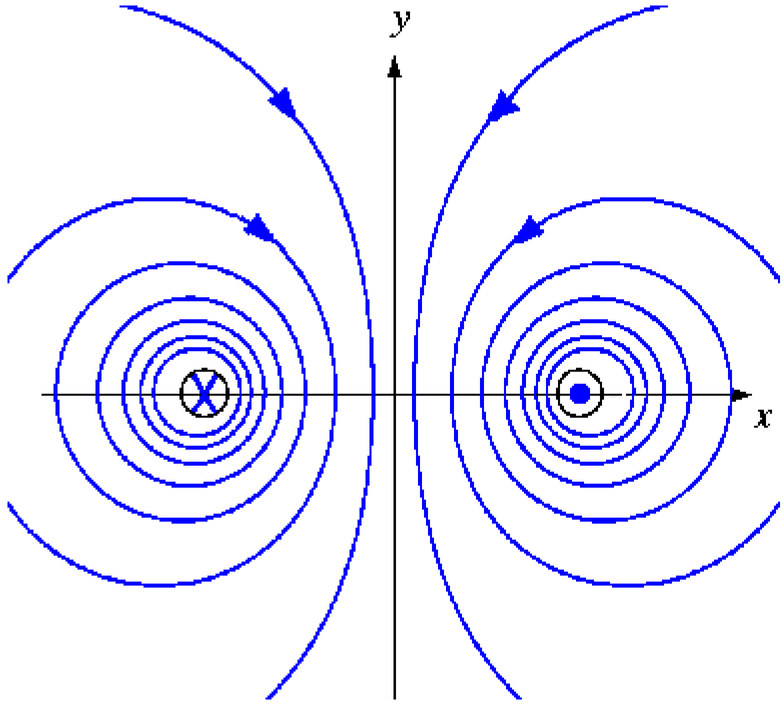


Figure Lines representing magnetic field - two parallel conductors carrying current in Z-direction and -Z-direction

See an animated model of the magnetic field from two conductors with current (results obtained from 2-dimentional electromagnetic modeling):
http://www.ele.auckland.ac.nz/~kacprzak/Model1/model1.htm

It was not until 1820 that a formal link was established between the sciences of **Electrostatics** and **Current Electricity** **and** **magnetism**. In that year Oersted discovered that a **magnetic compass needle was deflected in the neighborhood of an electric current – that the electric current produced a magnetic field.**

Within 3 months Ampere had developed a theory, which integrated the sciences of current electricity and magnetism. This theory is symbolized by the notion of **equivalence** of a **magnetic dipole** and a **current dipole**.

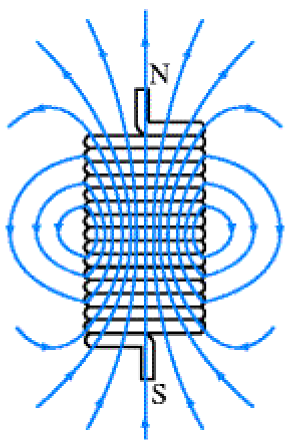


Figure Magnetic field from a solenoid

Consider now a bar magnet of pole strength Q_m and length L (magnetic dipole moment m= Q_m L) in a uniform field B. The north (+) pole experiences a force F= Q_m B to the right and the south (-) pole an equal force to the left.

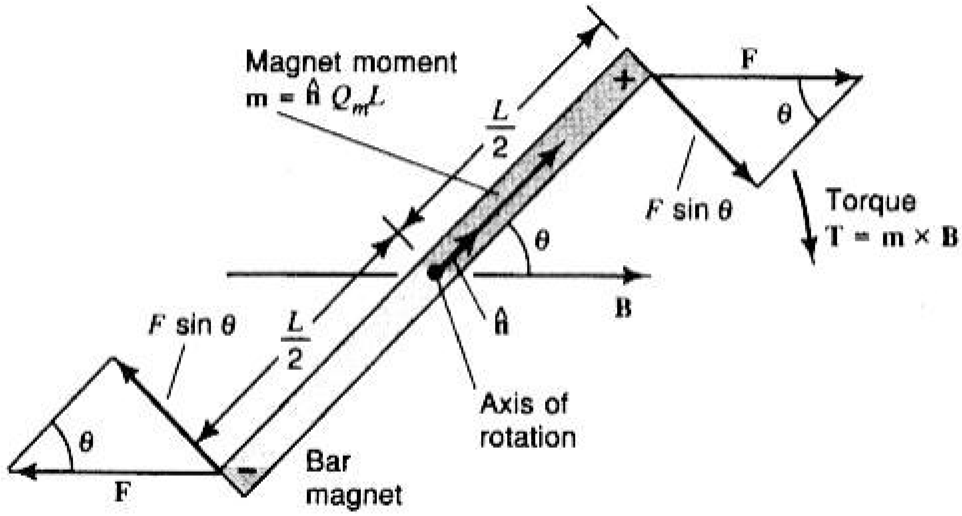


Figure Bar magnet experiences torque tending to align it with magnetic field **B**.
The torque T, or turning moment (force x distance), on the dipole is

$$T = 2F\frac{L}{2}\sin\theta$$

where: F= Q_m B, [N]

L = length of dipole, [m]

q = angle between dipole axis and B [rad/deg]

Including the value of F, we have

$$T = Q_m LB\sin\theta = mB\sin\theta$$

where: m=magnetic moment [Am^2]

In vector notation the torque is given by

$$T = m \times B = IA \times B$$

where: I = loop current, [A]

A = loop area, [m^2]

We note that the magnet and loop are equivalent if their magnetic moments are equal or

$$Q_m L = IA$$

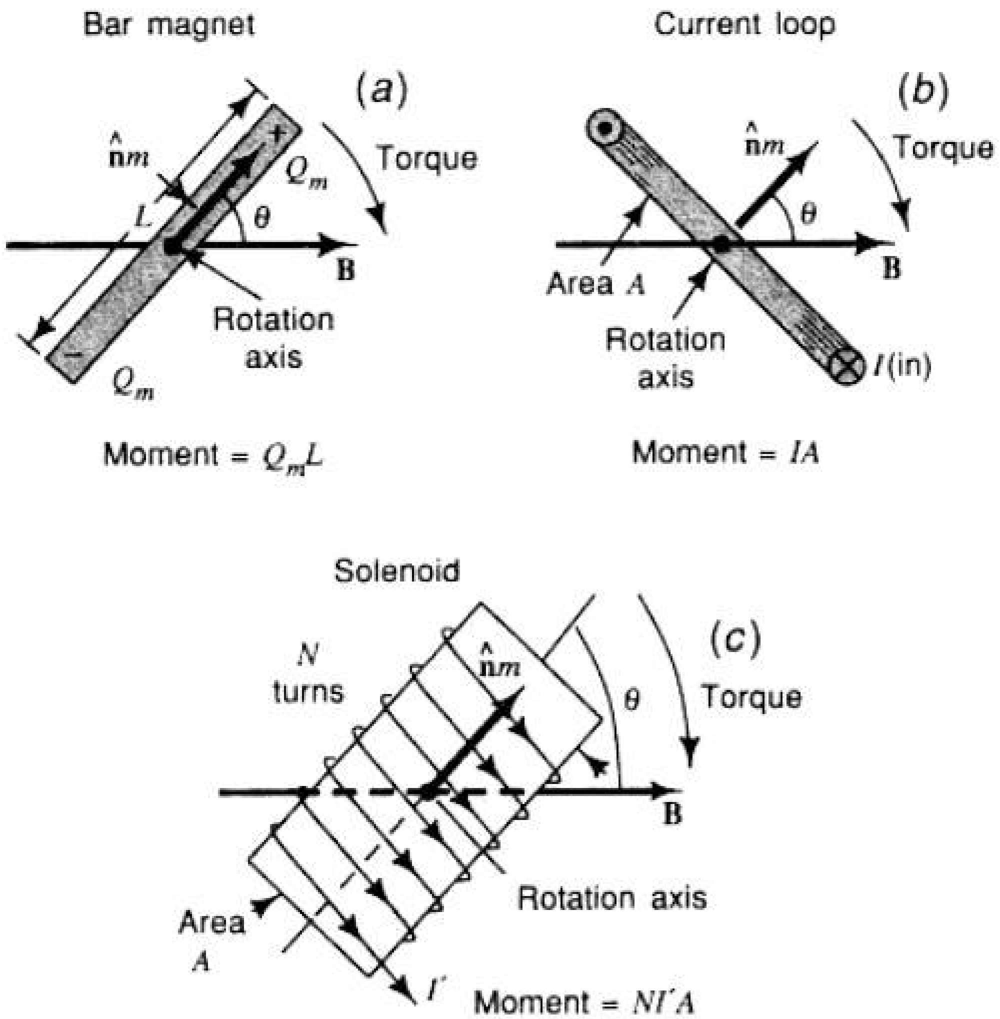


Figure Bar magnet (a), current loop (b), and solenoid (c) in a uniform field **B**. in all three cases the torque is clockwise and tends to align the magnetic moment **m** with **B.** If all moments are equal ($Q_m L=IA=NI`A$ [Am^2]) the torque is the same in all three cases. The loop is shown in cross-section.

The expression for the energy stored in a magnetic field is:

$$W = \frac{BH}{2}, \qquad (B = \mu H)$$

$$W = \frac{\mu H^2}{2} = \frac{B^2}{2\mu} \qquad \text{(If m is constant)}$$

The treatment of electricity and magnetism is from an **historically based** perspective, which makes it easier to discuss the magnetic properties of matter.

There are three important results, which are all totally consistent with each other:

1. Force on moving charges in a magnetic field

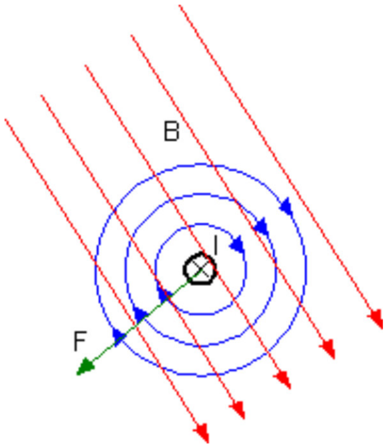


Figure Force on conductor in the magnetic field

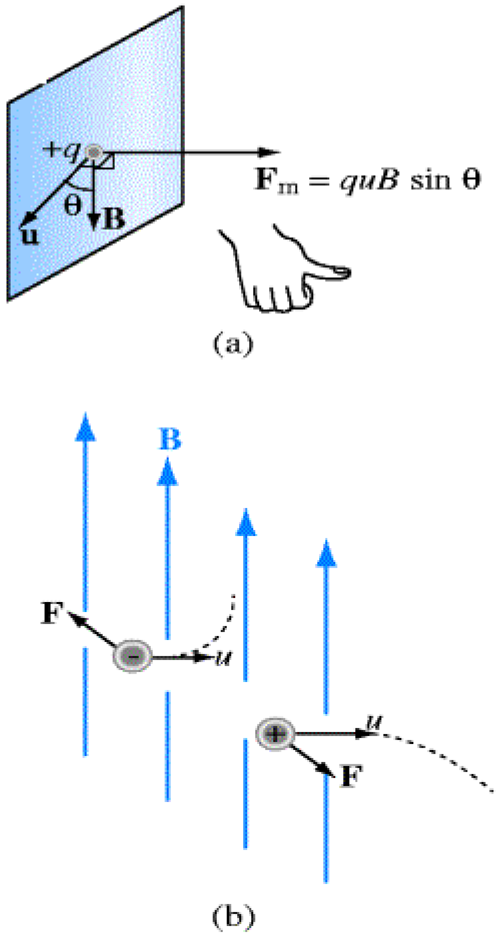
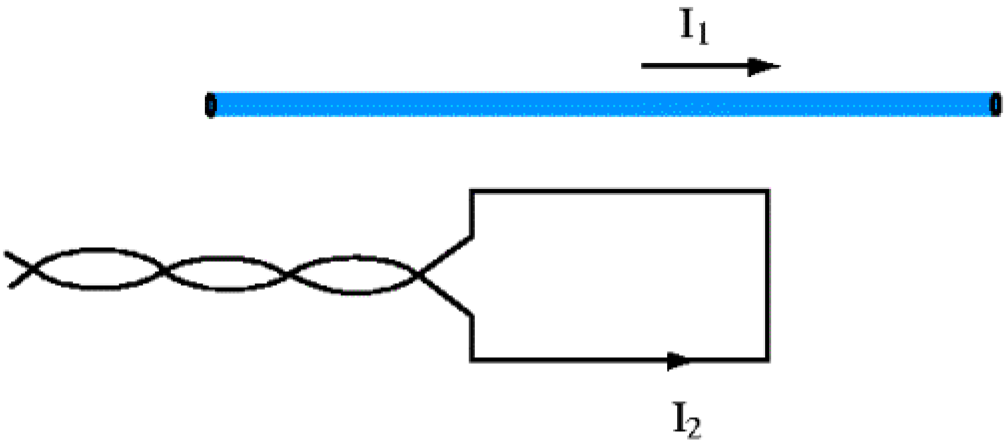


Figure The direction of the magnetic force exerted on a charged particle moving in a magnetic field is (a) perpendicular to both **B** and **u** and (b) depends on the charge polarity (positive or negative)

Problem: A wire loop



Given: A wire loop lies in the same plane as an infinitely long wire. Initially, neither wire is carrying a current.

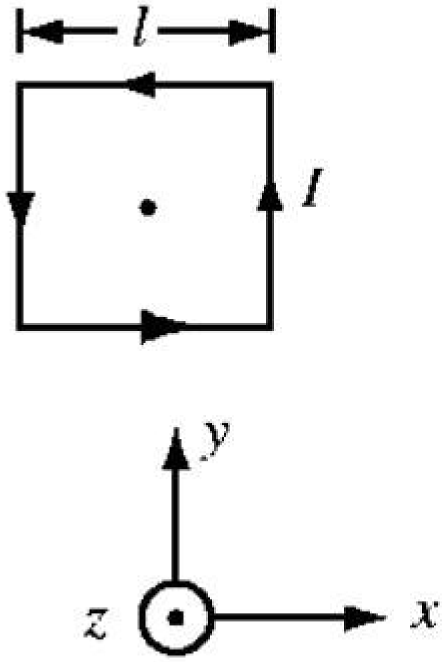
Q1. If I1=0 and a current I2 is made to flow through the loop in the direction shown, what will happen to the loop?

(a) Nothing.
(b) It will try to expand.
(c) It will contract.

Q2. If in addition to I2, a strong current I1 is made to flow through the linear wire, what is likely to happen to the loop?

(a) Nothing.
(b) It will try to expand.
(b) It will contract.

Problem: Field at Center of a Square



The magnetic flux density at a distance r from the midpoint of a conductor of length l is:

$$\mathbf{B} = \mu_0 \mathbf{H} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I l}{2\pi r \sqrt{4r^2 + l^2}} \qquad \text{(T)}.$$

Q. Use the above result to determine B at the center of a square of sides l.

(a) $\mathbf{B} = 0$

(b) $\mathbf{B} = \hat{\mathbf{z}}\, 2\sqrt{2}\mu_0 I/\pi l$

(c) $\mathbf{B} = -\hat{\mathbf{z}}\, 2\sqrt{2}\mu_0 I/\pi l$

(d) $\mathbf{B} = \hat{\mathbf{z}}\, \sqrt{2}\mu_0 I/2\pi l$

Solution:

There is only a normal component $\mathbf{B}_z$ in the center of the square. $\mathbf{B}_z$ is produced by the current flowing in all (four) sides l of the square.

Thus:

$$B = 4 \times \hat{z} \frac{\mu_0 I l}{2\pi \cdot r\sqrt{4r^2 + l^2}}$$

Because $r = \frac{l}{2}$,we get:

$$B = 4 \times \hat{z} \frac{\mu_0 I l}{2\pi \cdot \frac{l}{2}\sqrt{4\frac{l}{2^2}^2 + l^2}}$$

That leads to:

$$B = \hat{z} \frac{4\mu_0 I l}{\pi \cdot l\sqrt{l^2 + l^2}}$$

Thus

$$B = \hat{z} \frac{4\mu_0 I}{\pi \cdot l\sqrt{2}} = \hat{z} \frac{4\mu_0 I\sqrt{2}}{\pi \cdot l\sqrt{2} \cdot \sqrt{2}}$$

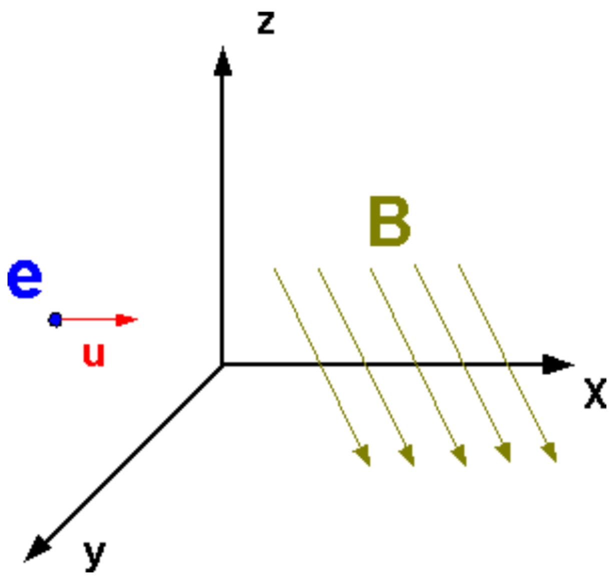
So the final result is

$$B = \hat{z} \frac{2\sqrt{2}\mu_0 I}{\pi \cdot l}$$

Problem 5-1 (Fundamentals of applied electromagnetics, F.T. Ulaby)

An electron with a speed of 4×10^6 m/s is projected along the positive x-direction into a medium containing a uniform magnetic flux density $B = (\hat{x}2 - \hat{z}3)$ T. Given that $e = 1.6 \times 10^{-19}$C and the mass of an electron is $m_e = 9.1 \times 10^{-31}$ kg, determine the initial acceleration vector of the electron (at the it is projected into the medium).

Solution



Variables known: m_e, e, u, B

Variable unknown: a=?

$$\mathbf{a} = \frac{\mathbf{F}}{m_e}$$

where

$$\mathbf{F} = e \cdot \mathbf{u} \times \mathbf{B}$$

The cross product is:

$$\mathbf{u}\times\mathbf{B} = \det\begin{vmatrix} 4\times10^6 & 0 & 0 \\ 2 & 0 & -3 \\ \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \end{vmatrix} = \begin{vmatrix} 4\times10^6 & 0 & 0 \\ 2 & 0 & -3 \\ \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \hdashline 4\times10^6 & 0 & 0 \\ 2 & 0 & -3 \end{vmatrix} =$$

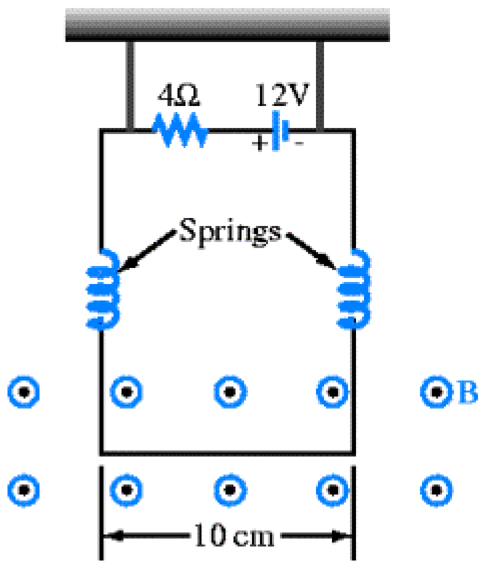
$$= 4\times10^6\times0\times\hat{\mathbf{z}} + 2\times\hat{\mathbf{y}}\times0 + \hat{\mathbf{x}}\times0\times(-3) +$$
$$-2\times0\times\hat{\mathbf{z}} - 4\times10^6\times\hat{\mathbf{y}}\times(-3) - \hat{\mathbf{x}}\times0\times0 = -\hat{\mathbf{y}}\,12\times10^6$$

And finally

$$\mathbf{a} = \frac{\mathbf{F}}{m_e} = \frac{1.6\times10^{-19}\times(-\hat{\mathbf{y}}\,12\times10^6)}{9.1\times10^{-31}} = -\hat{\mathbf{y}}\,2.1\times10^{18}\,(m/s^2)$$

Problem 5-3 (Fundamentals of applied electromagnetics, F.T. Ulaby)

The circuit shown in the figure uses two identical springs to support a 10-cm-long horizontal wire with a mass of 5g. In the absence of a magnetic field, the weight of the wire causes the springs to stretch a distance of 0.2 cm each. When a uniform magnetic field is turned on in the region containing the horizontal wire, the springs are observed to stretch an additional 0.5 cm. What is the intensity of the magnetic flux density **B**?



Without **B** – only gravitation force

$$F_g = mg = k\cdot\Delta L$$

where: k is springs coefficient, DL is change is a spring's length (0.2 cm)

Thus: $$k = \frac{mg}{\Delta L}$$

When **B** is applied:

$$F = F_g + F_B$$

Thus

$$F = mg + BIl = k \cdot \Delta L_2$$

and

$$k = \frac{mg + BIl}{\Delta L_2}$$

Next

$$\frac{mg}{\Delta L} = \frac{mg + BIl}{\Delta L_2}$$

Thus:

$$B = \left(\frac{mg \cdot \Delta L_2}{\Delta L} - mg \right) \Big/ Il$$

Finally

$$B = \left(\frac{0.005 \times 9.81 \times 0.007}{0.002} - 0.005 \times 9.81 \right) \Big/ \frac{12}{4} \cdot 0.1 = 0.408 \text{ T}$$

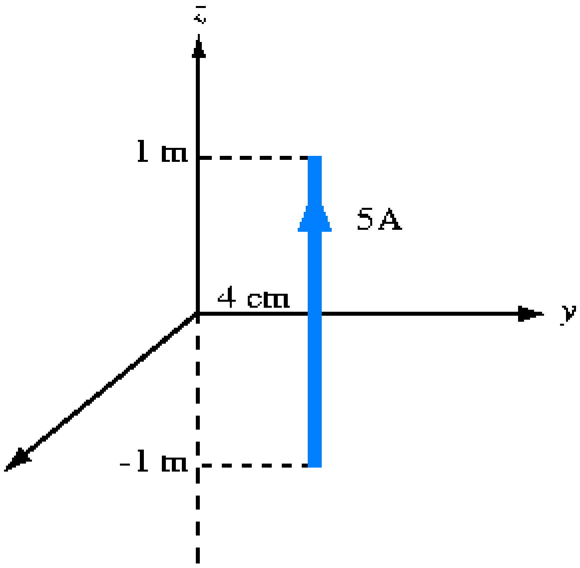


Figure P5.5: Problem 5.5.

Problem 5.5 In a cylindrical coordinate system, a 2-m-long straight wire carrying a current of 5 A in the positive z-direction is located at $r = 4$ cm, $\phi = \pi/2$, and -1 m $\le z \le 1$ m.

(a) If $\mathbf{B} = \hat{\mathbf{r}}\,0.2\cos\phi$ (T), what is the magnetic force acting on the wire?

(b) How much work is required to rotate the wire once about the z-axis in the negative ϕ-direction (while maintaining $r = 4$ cm)?

(c) At what angle ϕ is the force a maximum?

Solution:

(a)

$$\begin{aligned}\mathbf{F} &= I\boldsymbol{\ell}\times\mathbf{B}\\ &= 5\hat{\mathbf{z}}2\times[\hat{\mathbf{r}}0.2\cos\phi]\\ &= \hat{\phi}\,2\cos\phi.\end{aligned}$$

At $\phi = \pi/2$, $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}$. Hence,

$$\mathbf{F} = -\hat{\mathbf{x}}\,2\cos(\pi/2) = 0.$$

(b)

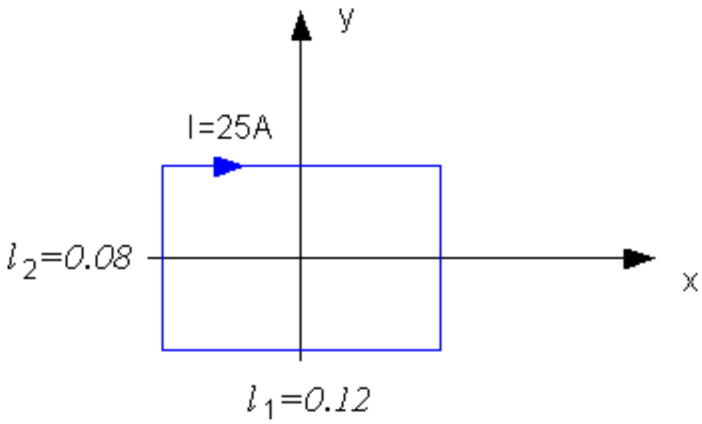
$$\begin{aligned}W &= \int_{\phi=0}^{2\pi}\mathbf{F}\cdot d\mathbf{l} = \int_0^{2\pi}\hat{\boldsymbol{\phi}}[2\cos\phi]\cdot(-\hat{\boldsymbol{\phi}})r\,d\phi\Big|_{r=4\text{ cm}}\\ &= -2r\int_0^{2\pi}\cos\phi\,d\phi\Big|_{r=4\text{ cm}} = -8\times10^{-2}[\sin\phi]_0^{2\pi} = 0.\end{aligned}$$

The force is in the $+\hat{\phi}$-direction, which means that rotating it in the $-\hat{\phi}$-direction would require work. However, the force varies as $\cos\phi$, which means it is positive when $-\pi/2 \leq \phi \leq \pi/2$ and negative over the second half of the circle. Thus, work is provided by the force between $\phi = \pi/2$ and $\phi = -\pi/2$ (when rotated in the $-\hat{\phi}$-direction), and work is supplied for the second half of the rotation, resulting in a net work of zero.

(c) The force **F** is maximum when $\cos\phi = 1$, or $\phi = 0$.

Problem 5-7 (Fundamentals of applied electromagnetics, F.T. Ulaby)

An 8 cm X12 cm rectangular loop of wire is situated in the x-y plane with the center of the loop at the origin and its long sides parallel to the x-axis. The loop has a current of 25A flowing clockwise (when viewed from above). Determine the magnetic field at the center of the loop.



Solution

$$H = 2H_1 + 2H_2$$

where

$$H_1 = \frac{I \cdot l_1}{2\pi \cdot r_1\sqrt{4r^2 + l_1^{\,2}}} = \frac{25 \cdot 0.12}{2\pi 0.04\sqrt{4(0.04)^2 + (0.12)^2}} = 83.33 \quad (A/m)$$

and

$$H_2 = \frac{I \cdot l_2}{2\pi \cdot r_2\sqrt{4r^2 + l_2^{\,2}}} = \frac{25 \cdot 0.08}{2\pi 0.06\sqrt{4(0.06)^2 + (0.08)^2}} = 36.83 \quad (A/m)$$

Thus

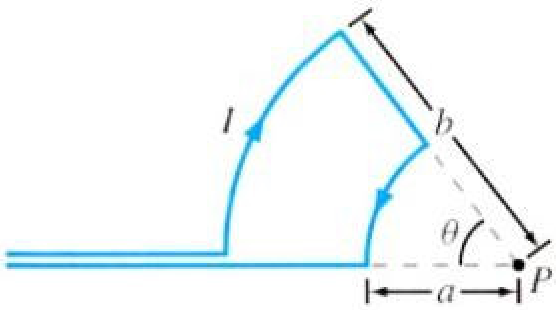
$$H = 2H_1 + 2H_2 = 240.32 \ (A/m)$$

and magnetic flux density is

$$B = \mu_0 H = 4\pi \times 10^{-7} \times 240.32 = 3 \times 10^{-4} \ \ (T)$$

Problem 5-9 (Fundamentals of applied electromagnetics, F.T. Ulaby)

The loop shown in the figure consists of radial lines and segments of circles whose centers are at point P. Determine the magnetic field H at P in terms of a,b,q and I.



Basically we deal with two arcs carrying current I. The lengths of the arcs are: L1=qa and L2=qb

In the center of a circle:

$$H = \hat{z}\frac{I}{4\pi r}$$

where: r is the radius of a circle

as

$$a = \frac{L_1}{\theta}$$

and

$$b = \frac{L_2}{\theta}$$

The magnetic filed from the arcs are:

$$H_1 = \hat{z}\frac{I\theta}{4\pi a}$$

and

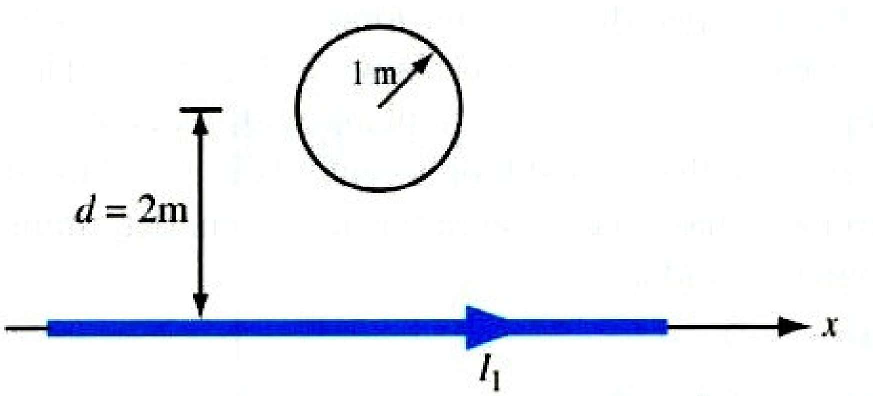
$$H_2 = \hat{z}\frac{I\theta}{4\pi b}$$

Finally

$$H = H_1 - H_2 = \hat{z}\frac{I\theta}{4\pi a} - \hat{z}\frac{I\theta}{4\pi b} = \hat{z}\frac{I\theta}{4\pi}\left(\frac{1}{a} - \frac{1}{b}\right) = \hat{z}\frac{I\theta}{4\pi}\left(\frac{b-a}{ab}\right)$$

Problem 5-11 (Fundamentals of applied electromagnetics, F.T. Ulaby)

An infinitely long wire carrying a 50-A current in the positive x-direction is placed along the x-axis in the vicinity of a 10-turn circular loop located in the x-y plane as shown in figure. If the magnetic field at the center of the loop is zero, what is the direction and magnitude of the current flowing in the loop?



Solution

Magnetic field generated by the current in the loop is

$$H_1 = N\frac{I_{loop}}{2a}$$

Magnetic field from generated by the current in the conductor is

$$H_2 = \frac{I}{2\pi d}$$

If the magnetic field at the center of the loop is zero

$$|H_1| = |H_2|$$

Thus

$$N\frac{I_{loop}}{2a} = \frac{I}{2\pi d}$$

So

$$I_{loop} = \frac{Ia}{\pi r N} = \frac{50 \cdot 1}{3.14 \cdot 2 \cdot 10} = 0.79 \quad (A)$$

Problem 5-13 (Fundamentals of applied electromagnetics, F.T. Ulaby)

A long East-West-oriented power cable carrying an unknown current I is at a height of 8 m above the Earth's surface. If the magnetic flux density recorded by a magnetic-field meter placed at the surface is 12×10^{-6} T when the current is flowing through the cable and 20×10^{-6} T when the current is zero, what is the magnitude of I?

Solution

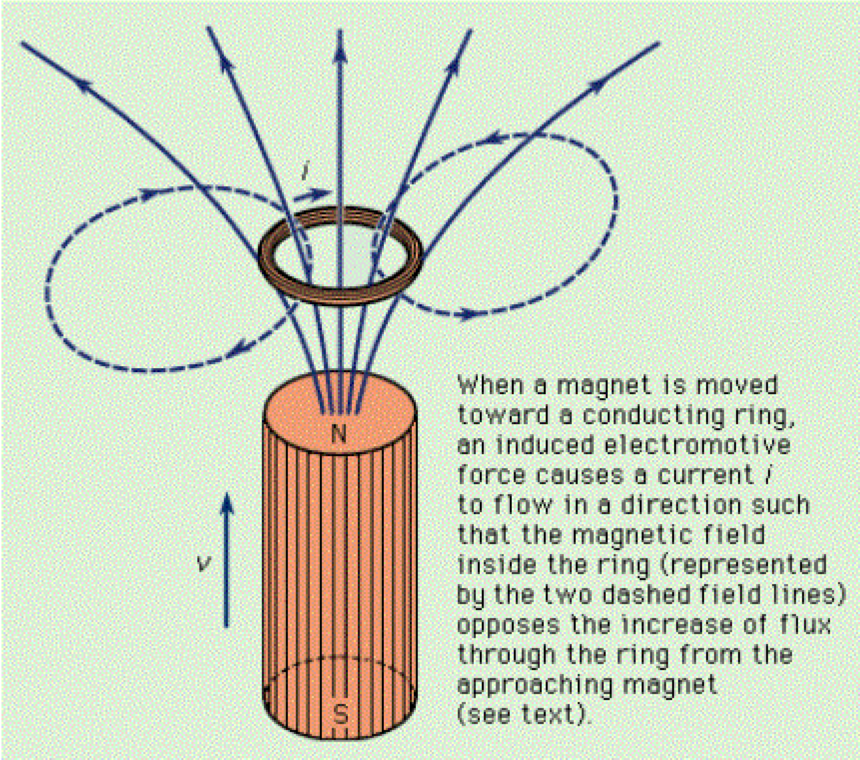
Basically:

$$B = \frac{\mu I}{2\pi r}$$

Thus

$$I = \frac{B 2\pi r}{\mu_0} = \frac{(20-12)\times 10^{-6} \times 2 \times \pi \times 8}{4\pi \times 10^{-7}} = 320 \ (A)$$

2. Faraday's Law



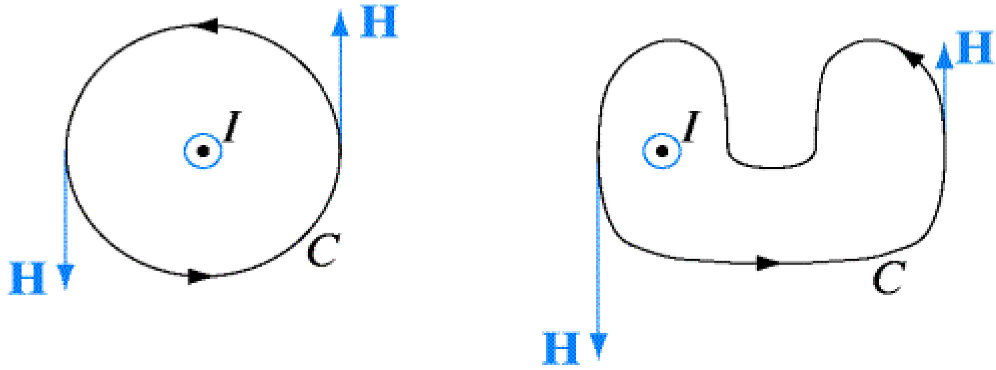
Any change in the magnetic environment of a coil of wire will cause a voltage (emf) to be "induced" in the coil. No matter how the change is produced, the voltage will be generated. The change could be produced by changing the magnetic field strength, moving a magnet toward or away from the coil, moving the coil into or out of the magnetic field, rotating the coil relative to the magnet, etc.

$$v = -Z\frac{d\Phi}{dt}$$

where: v - electromotive force, Z- number of turns in the coil,
t – time,F magnetic flux

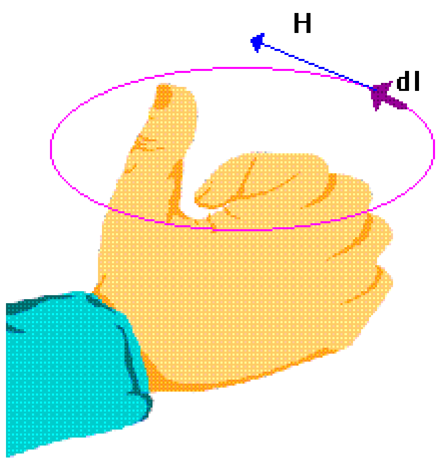
3. Ampere's law

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I$$



Ampere's circular law states that the line integral of **H** around a closed path is equal to the current traversing the surface bounded by the path.

The sign convention for the direction of **C** is taken so that **I** and **H** satisfy the right-hand rule.



The right hand rule: Curl your right-hand fingers around the closed path (Amperian loop), with them pointing in the direction of integration. A current passing through the loop in the general direction of your outstretched thumb is assigned a plus sign, and a current in the opposite direction is assigned a minus sign.

This course will be based completely on these laws so that the working understanding of how electromagnetic devices work is ultimately traceable to results, which have been developed experimentally.

The Magnetic Properties of Matter

In vacuum the magnetic flux density **B** and the magnetic field intensity **H** are related by the expression

$$\mathbf{B} = m_0\,\mathbf{H} \quad \text{where } m_0 = 4p \times 10^{-7}\ \text{H/m}$$

Where the field exists within a medium however the expression is modified to the general expression

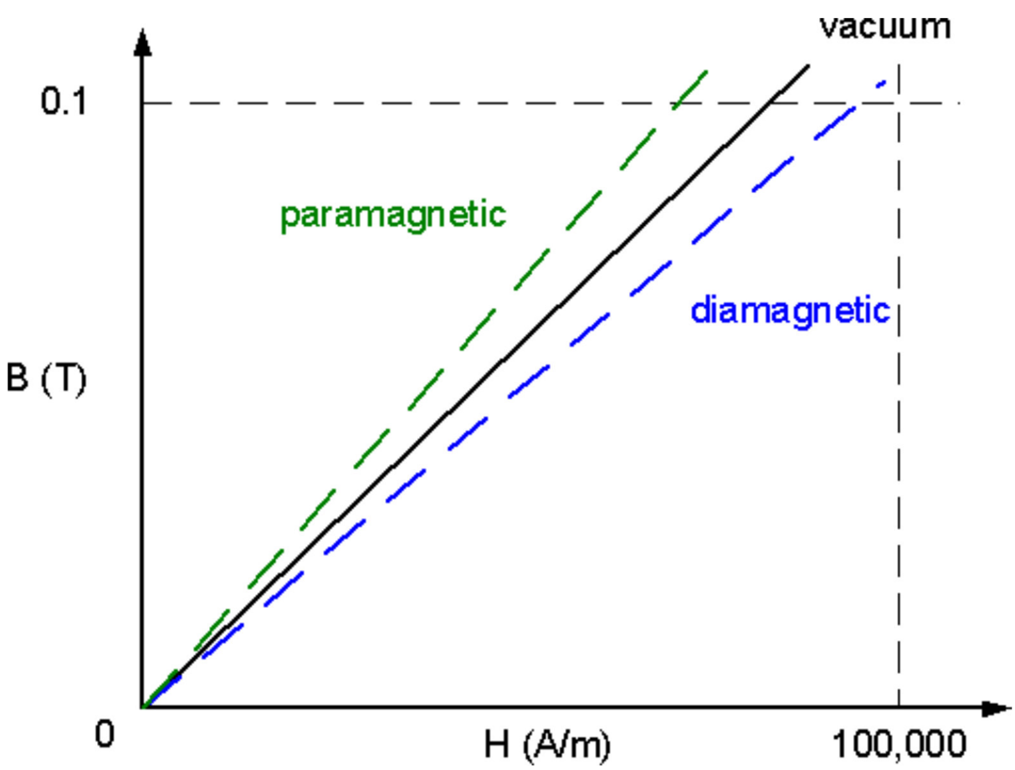
$$\mathbf{B} = m.\mathbf{H} = m_0 m_r\,\mathbf{H}$$

Where m_r is the relative permeability of the medium. This behaviour is caused by the existence of **magnetic dipole moments** within the atoms caused by spin and orbital motion of electrons in those atoms.

For paramagnetic materials (eg oxygen) and diamagnetic materials (eg carbon) m_r is a constant and very close to unity:

Oxygen (20^0 C) $\quad m_r = 1 + (1330 \times 10^{-6})$

Carbon (graphite) $\quad m_r = 1 - (45 \times 10^{-6})$



Paramagnetism occurs primarily in substances in which some or all of the individual atoms, ions, or molecules possess a permanent magnetic dipole moment.

The atoms in diamagnetic material have no permanent magnetic dipole moments.

A ferromagnetic substance contains permanent atomic magnetic dipoles that are spontaneously oriented parallel to one another even in the absence of an external field.

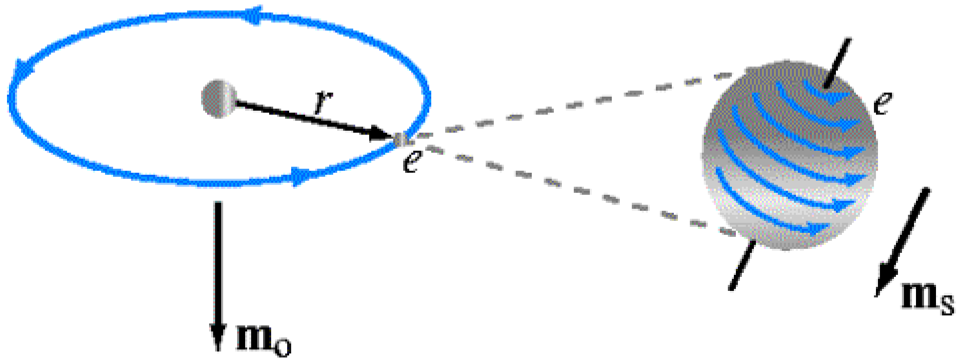
In ferromagnetic material m_r may attain values as high as 10^5 - 10^6 **but it is not constant** and varies with **H** in quite a complex way.

Ferromagnetic materials are also of considerable practical importance and will be covered later in the course.

Atomic Magnetic Dipole Moments

The simple model of the atom consists of a nucleus surrounded by a cloud of electrons in nominally specified orbits. Each electron has a charge of 1.6×10^{-19} C and this charge may be considered to be concentrated in a small sphere. In addition to orbiting the nucleus each electron also spins on its own axis while it moves along its orbit. This movement creates dipole moments – there are two, the **orbital** and **spin** dipole moments denoted by m_o and m_s respectively. m_s has a magnitude of 9.27×10^{-24} Ampere**x**metres^2 , and m_o is either zero or an integral multiple of the value of m_s.

In the atoms of many elements the electrons are arranged symmetrically so that the magnetic moments due to the spin and orbital motion cancel out, leaving the atom with zero magnetic moment. However the atoms of more than 1/3rd of known elements lack this symmetry so that they (the atoms) do possess a magnetic moment. However in most of these materials the arrangement of the atoms is such that the magnetic moment of one is cancelled out by that of an oppositely directed near neighbour.



(a) Orbiting electron (b) Spinning electron

An electron with charge of **–e** moving with a constant velocity **u** in a circular orbit of radius **r** completes one revolution in time

$$T = \frac{2\pi r}{u}$$

This circular motion of the electron constitutes a tine current loop with current **I** given by:

$$I = -\frac{e}{T} = -\frac{eu}{2\pi r}$$

The magnitude of the associated **orbital magnetic moment** m_0 is

$$m_0 = IA = -\frac{eu}{2\pi \cdot r} \cdot \pi r^2 = -\frac{eur}{2} = -\left(\frac{e}{2m_e}\right)L_e$$

where: $\mathbf{L_e = m_e ur}$ is the angular momentum of the electron and $\mathbf{m_e}$ in its mass.

The magnitude of $\mathbf{m_s}$ (spin magnetic moment) predicted by quantum theory is

$$m_s = -\frac{eh}{2m_e}$$

where: ħ is Planck's constant.

Only in five elements are the atoms arranged with their magnetic moments in parallel so that they supplement rather than cancel one another. **These are the ferromagnetic elements: iron, nickel, cobalt, dysprosium, and gadolinium.**

Do you know?

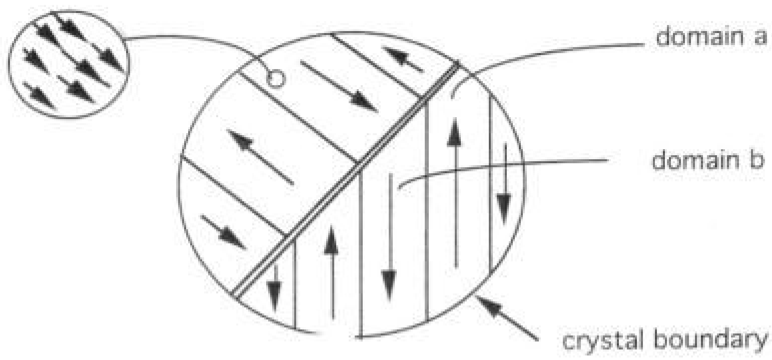
Magnetic resonance (MR) imaging is founded on the manipulation of magnetic dipole moments in such a way that signals generated from these interactions that can be translated into visual images of the body.

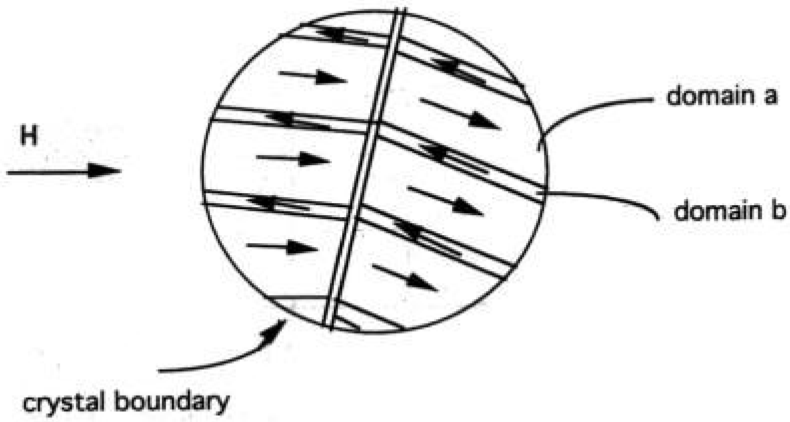
Figure Typical MR images of the head (left), neck (middle) and kidneys (right).

Ferromagnetism

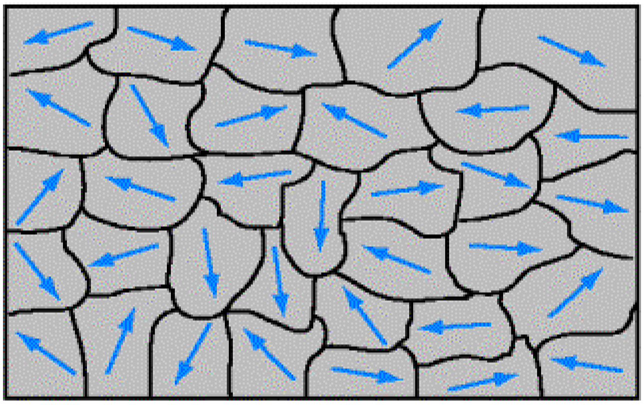
In addition to the above 5 elements a number of their alloys including non-ferromagnetic elements in their composition may also possess the property of ferromagnetism. Ferromagnetic materials are characterised in that they have a crystal structure divided into magnetic domains usually of microscopic size, in each of which the magnetic moments of the atoms are aligned. The alignment direction differs however from one domain to another.

The direction of alignment of the magnetic moments is normally along one of the crystal axes but the domains may be oriented randomly in three directions, and a single crystal may contain many domains.

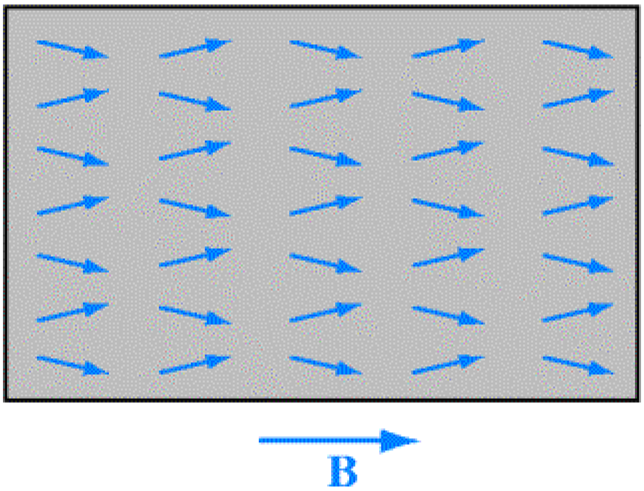




In the presence of an external **H** field favourably oriented domains **increase** in size through 'domain wall motion'. Also the atomic dipole moments tend to "**rotate"** into alignment with **H**.



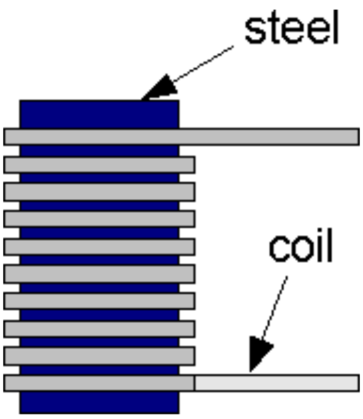
(a) Unmagnetized domains



(b) Magnetized domains

Figure Comparison of (a) unmagnetised and (b) magnetised domains in a ferromagnetic material

The magnetisation process



Assume a (non-magnetised) cast steel specimen inside a magnetizing coil. As the coil current is increased from zero (increasing **H**) **B** increases as shown.

It is convenient to think of **B** as made up of two components: $\mathbf{B}=\mathbf{B}_0+\mathbf{B}_m$

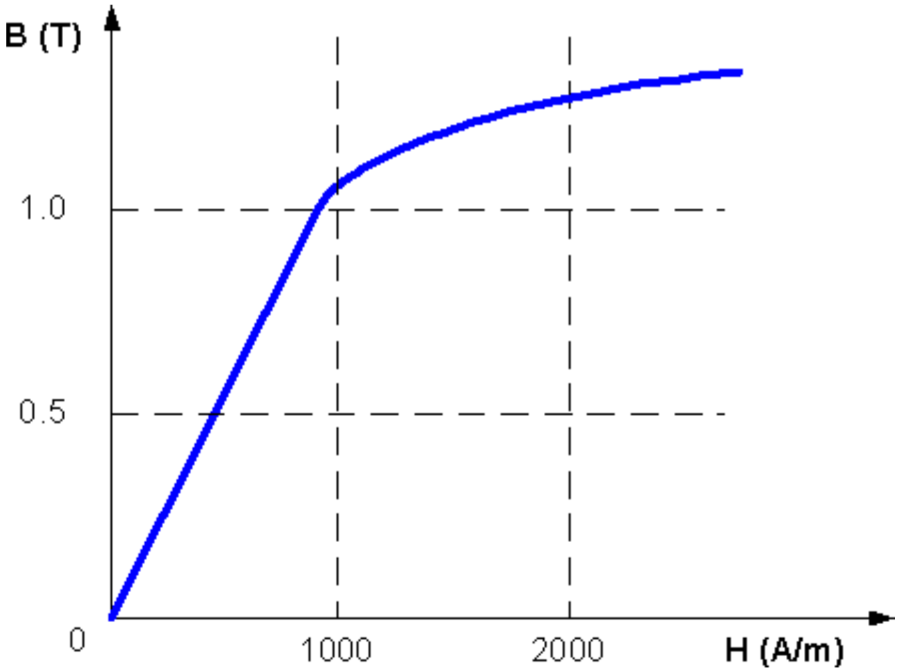
Where: $\mathbf{B}_0$ in the flux density due to the coil in a vacuum

And $\mathbf{B}_m$ in the extra flux density due to the material

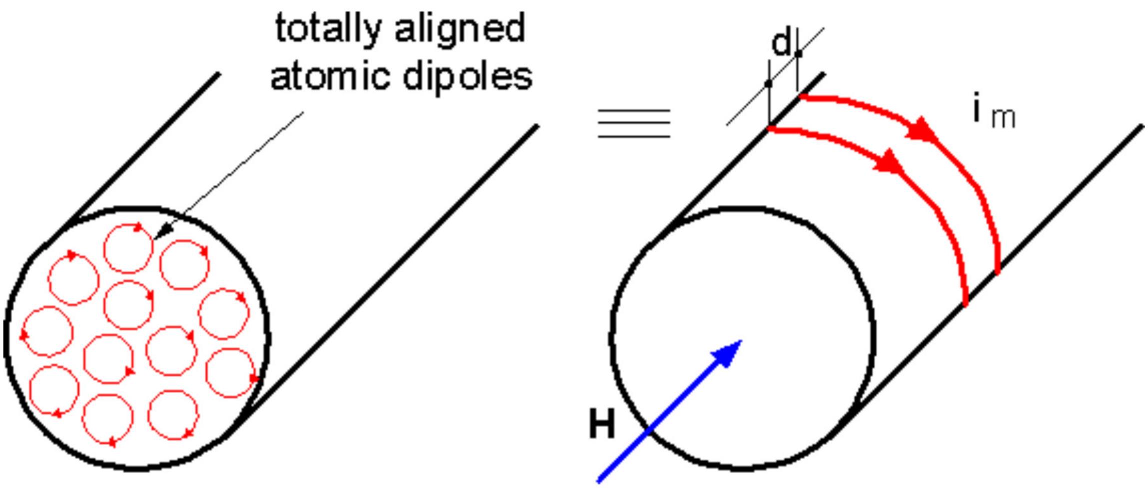
At **H**=1000A/m $\mathbf{B_0}$=4px10^{-4}=0.00125 T.

From the graph for the cast steel $\mathbf{B_m}$ »1.2 – $\mathbf{B_0}$ »1.2T.

Note that $\mathbf{B_m}$ >> $\mathbf{B_0}$, which is typical for ferromagnetic materials.



$\mathbf{B_m}$ is a result of the partial alignment of the magnetic moments with **H.**



A cross section of the specimen may be represented as shown. The effects of the adjacent currents cancel and it is only on the outer surface that there is no cancellation and here the effect is identical to a fictitious surface current i_m.

Each atom of iron has a magnetic moment 2.2 times the basic spin quantum (9.27 x 10^{-24} A.m^2). The spacing between atoms of iron (cubic crystal) is d = 2.27 x

10^{-10}m. The area occupied by a single atom in a cross section is d^2 so that for one atom

$$P_m = i_m . d^2 = 2.2 \times 9.27 \times 10^{-24} \; (A.m^2)$$

and

$$i_m = \frac{2.2 \cdot 9.27 \times 10^{-24}}{2.27^2 \times 10^{-20}} = 396 \times 10^{-6} \;\; A$$

Successive layers of atoms are spaced *d* apart. The equivalent magnetic field intensity due to fictitious surface current will be

$$H_{equiv.} = \frac{i_m}{d} = \frac{i_m d^2}{d^3} = \frac{P_m}{d^3} \quad (A/m)$$

Since d^3 is the volume of an atom, **H_{equiv} is the maximum possible magnetic moment per unit volume of iron** (= magnetisation **M**).

Thus for iron

$$M_{max} = H_{equiv.} = \frac{394 \times 10^{-6}}{2.27 \times 10^{-10}} = 1.73 \times 10^{6} \quad (A/m)$$

The flux density produced in vacuum by $\mathbf{H_{equiv.}}$ is

$$B_{Mmax} = \mu_0 M_{max.} = 4\pi \times 10^{-7} \times 1.73 \times 10^{6} = 2.18 \;\; T$$

$\mathbf{B}_{Mmax}$ (=2.18Tesla) is the maximum component of flux density, which can be induced inside a ferromagnetic (iron) specimen due to the total alignment of all the atomic magnetic dipoles – achievement of this state corresponds to the **complete saturation** of the iron.

The completeness of saturation and the ease with which it is approached depend on the material – in particular the elements that have been alloyed with the iron and the physical and heat treatment of the specimen.

The magnetic vector **M** of a material is defined as the vector sum of the magnetic dipole moments of the atoms contained in a unit volume of the material.

The magnetic flux density corresponding to **M** is $\mathbf{B_m}=m_0\mathbf{M}$.

In the presence of an extermally applied magnetic field **H**, the total magnetic flux density in the material is

$$\mathbf{B}=m_0\mathbf{H}+m_0\mathbf{M}=m_0(\mathbf{H}+\mathbf{M})$$

where the first term represents the contribution of the external field and the second term represents the contribution of the magnetization of the material.

In general, a material becomes magnetized in response to the external field H, hence M can be explained as

$$\mathbf{M}=c_m\mathbf{H}$$

Where c_m is a dimensionless quantity called the magnetic susceptibility of the material.

For diamagnetic and paramagnetic materials, c_m is constant at a given temperature, resulting in a linear relationship between **M** and **H**.

For ferromagnetic substances the relationship between **M** and **H** not only is nonlinear, but also depends on the previous “history” of the material.

If

$$\mathbf{B}=m_0(\mathbf{H}+\mathbf{M})= m_0(\mathbf{H}+c_m\mathbf{H})= m_0(1+c_m)\mathbf{H}$$

and

$$\mathbf{B}=m\mathbf{H}$$

than

$$m=m_0(1+c_m) \quad (\mathrm{H/m})$$

Often it is convenient to define the magnetic properties of a material in terms of the relative permeability m_r

$$\mu_r = \frac{\mu}{\mu_0} = 1 + \chi_m$$

	Diamagnetism	Paramagnetism	Ferromagnetism
Typical value of c_m	» -10^{-5}	» 10^{-5}	½c_m ½>>1 and hysteretic
Typical value of m_r	»1	»1	½m_r ½>>1 and hysteretic

Problem 5-29 (Fundamentals of applied electromagnetics, F.T. Ulaby)

Iron contains 8.5 x 10^{28} atoms/m^3. At saturation, the alignment of the electrons' spin magnetic moments in iron can contribute 1.5 T to the total magnetic flux density **B**. If the spin magnetic moment of a single electron is 9.27 x 10^{-24} (Am^2), how many electrons per atom contribute to the saturated field?

Solution

$$B = \mu_0 H + \mu_0 M$$

The contribution to **B** is represented by $\mu_0 M$.
Let's write:

$$\mu_0 M = \Delta B$$

and

$$M = \frac{\Delta B}{\mu_0} = \frac{1.5}{4\pi \times 10^{-7}} = 1994267.5 \quad (A/m)$$

Magnetisation is the sum of spin magnetic moment (m_e) coming from all electrons (n).

$$M = n \cdot m_e$$

Thus

$$n = \frac{M}{m_e} = \frac{1994267.5}{9.27\times10^{-24}} = 1.28\times10^{29} \quad (electrons)$$

The number of electrons per atom is:

$$n_a = \frac{n}{N} = \frac{1.28\times10^{29}}{8.5\times10^{28}} = 1.5 \quad (electron/\ atom)$$

where N is the number of atoms in one cubic meter.

Magnetic Hysteresis of Ferromagnetic Materials

Common to all ferromagnetic materials is a characteristic feature described by magnetized domains. A magnetized domain of a material is a microscopic region (on the order of 10^{-10}m^3) within which the magnetic moments of all its atoms (typically on the order of 10^{19} atoms) are aligned parallel to each other.

In the absence of an external magnetic field, the domains take on random orientations relative to each other resulting in a net magnetization of zero.

The domain walls forming the boundaries between adjacent domains consist of thin transition regions.

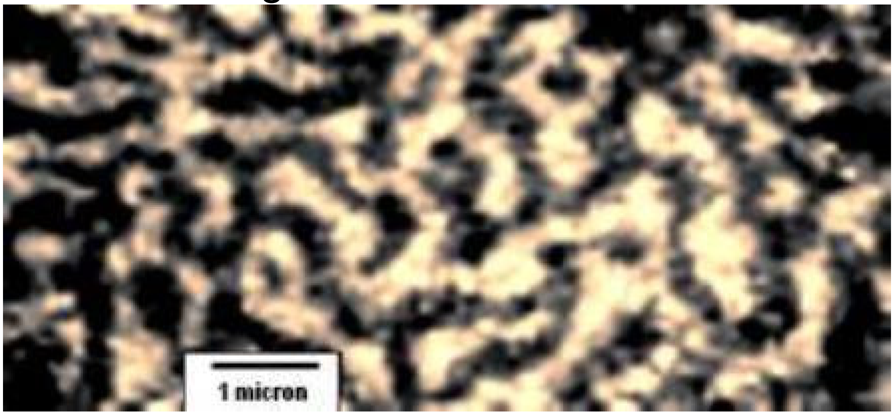


Figure Image of magnetic domains: Image of magnetic domains in a Pt/Co multilayer obtained with the scanning aperture photoemission microscope.

When an unmagnetized sample of a ferromagnetic material is placed, the domains will align partially with the external field.

This magnetization behavior of a ferromagnetic material is described in terms of **B-H magnetization curve**.

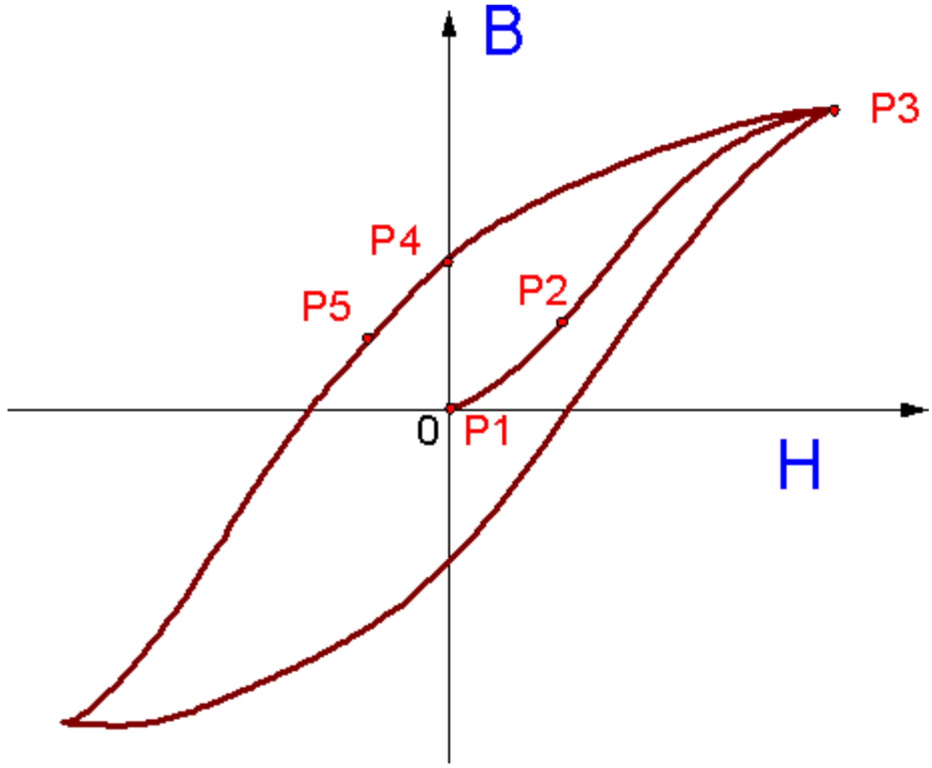


Figure Typical hysteresis curve for a ferromagnetic material

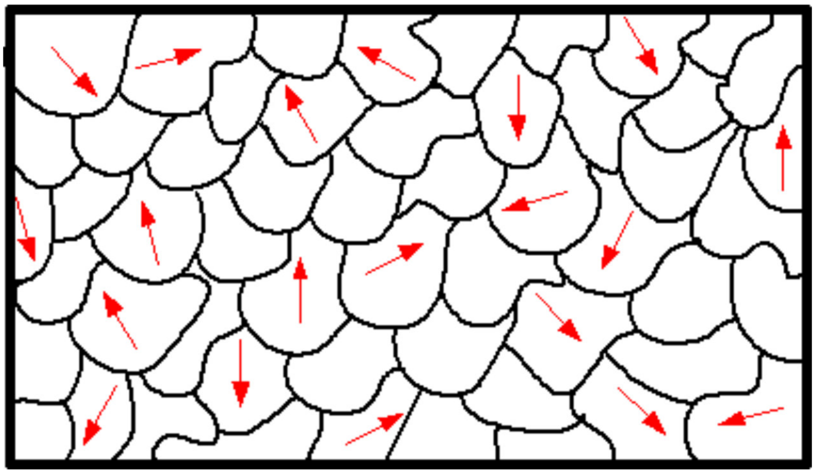


Figure. Domains in P1

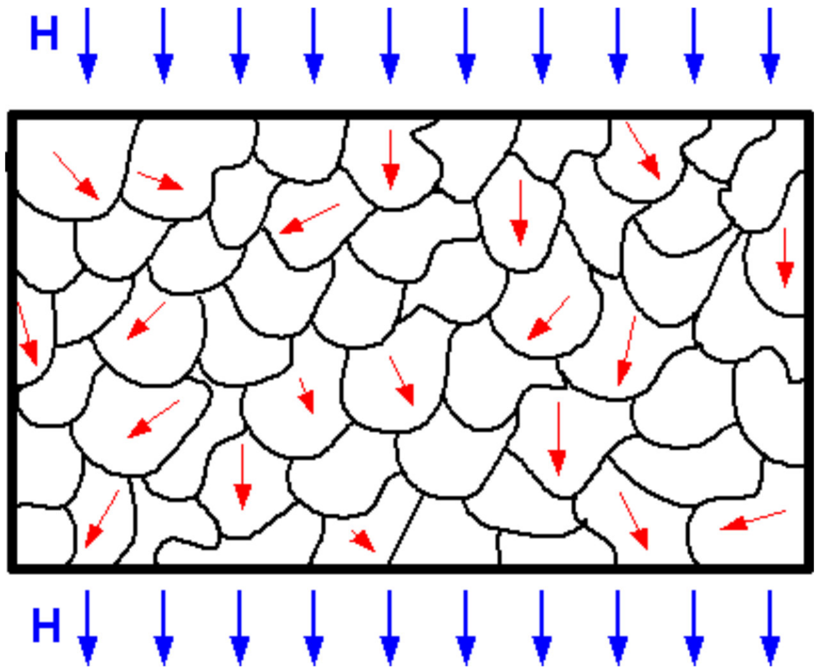


Figure. Domains in P2

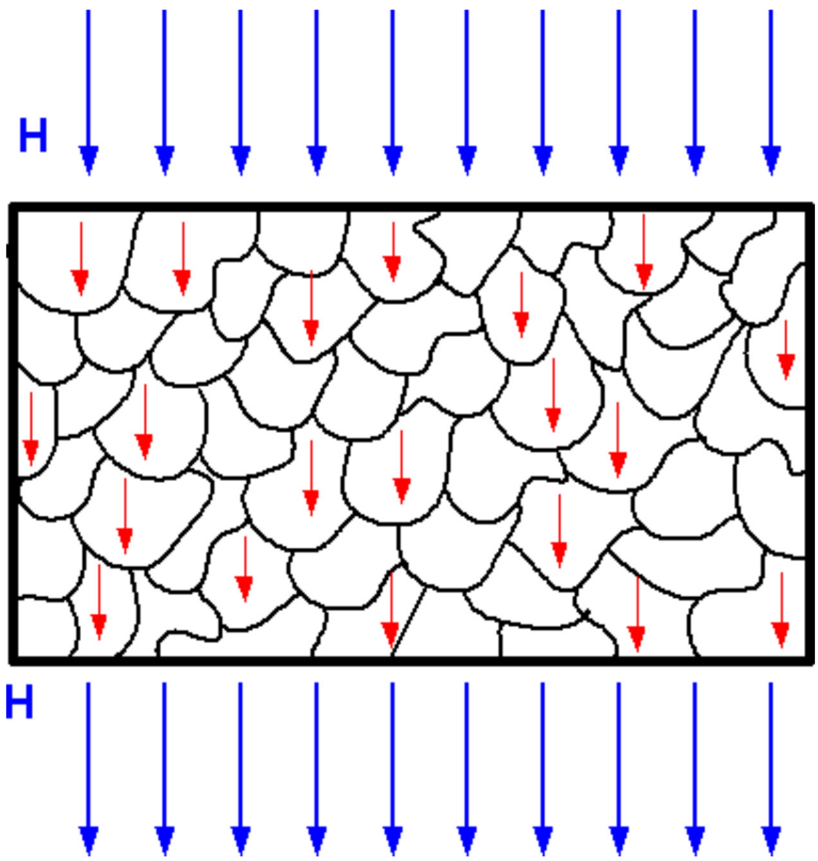


Figure. Domains in P3

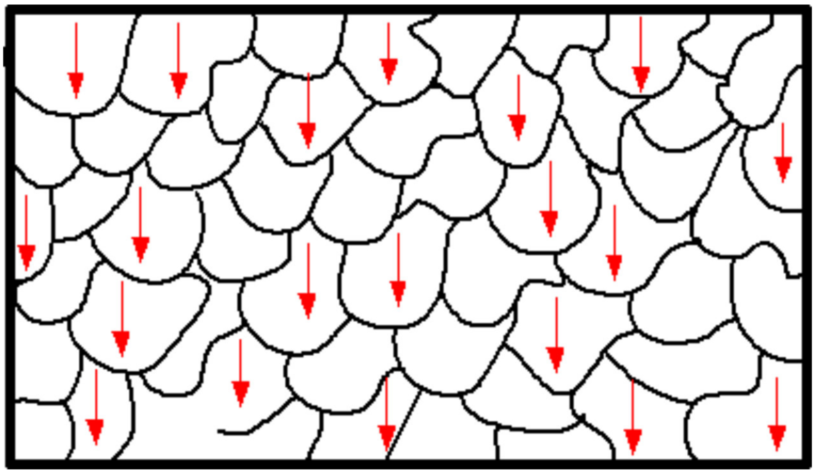


Figure. Domains in P4

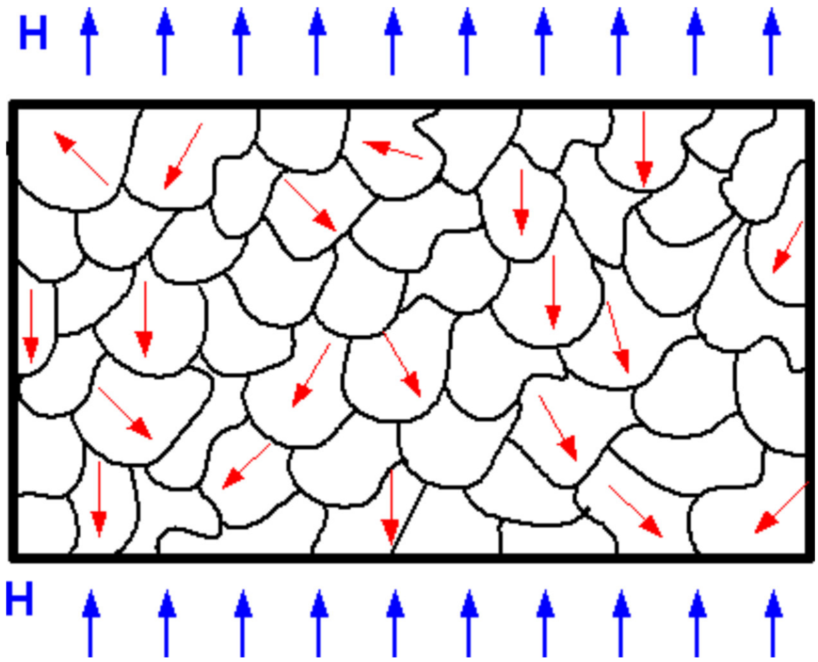
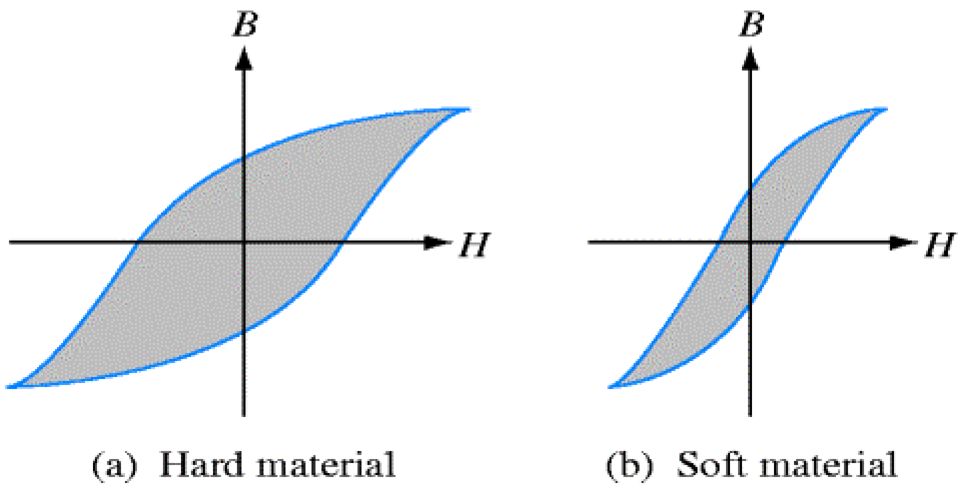


Figure. Domains in P5

The hysteresis loop shows that the magnetization process in ferromagnetic materials depends not only on the external magnetic field **H**, but on the magnetic history of the material as well.

Materials characterized by wide hysteresis loops are called hard magnetic materials. These materials cannot be easy demagnetized. Hard ferromagnetic materials are used in the fabrication of permanent magnets for motors and generators. Soft magnetic materials have narrow hysteresis loops.



Magnetic Boundary Conditions

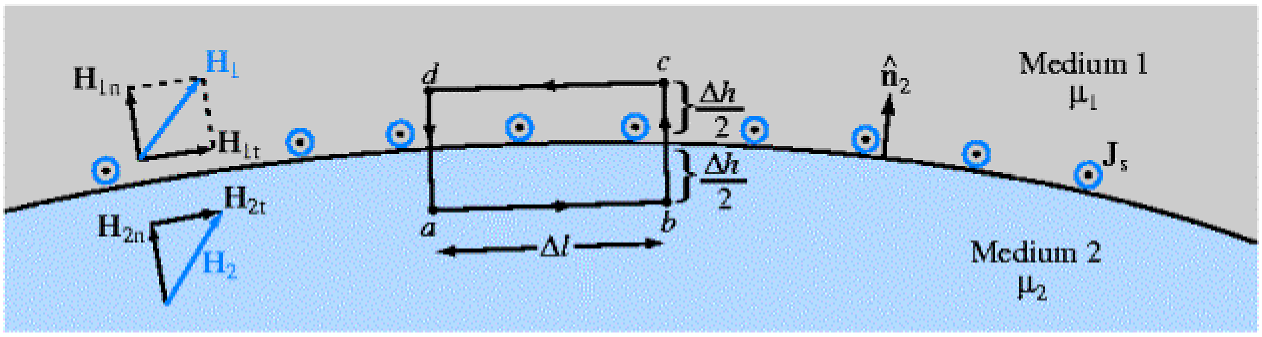


Figure Boundary between medium1 and medium 2

The normal component of **B** is continuous across the boundary between two adjacent media. In view of relations

$$B_{1n} = B_{2n}$$

Thus

$$\mu_1 \cdot H_{1n} = \mu_2 \cdot H_{2n}$$

To obtain the boundary condition for the tangential component of H, the following analysis is performed.

$$\int_C H \cdot dl = I$$

$$\int_C H \cdot dl = \int_a^b H_2 \cdot dl + \int_c^d H_1 \cdot dl = I$$

If h=0 (see the figure):

$$H_{2n} \cdot \Delta l - H_{2n} \cdot \Delta l = J_s \cdot \Delta l$$

where J_s – surface current density (normal component)

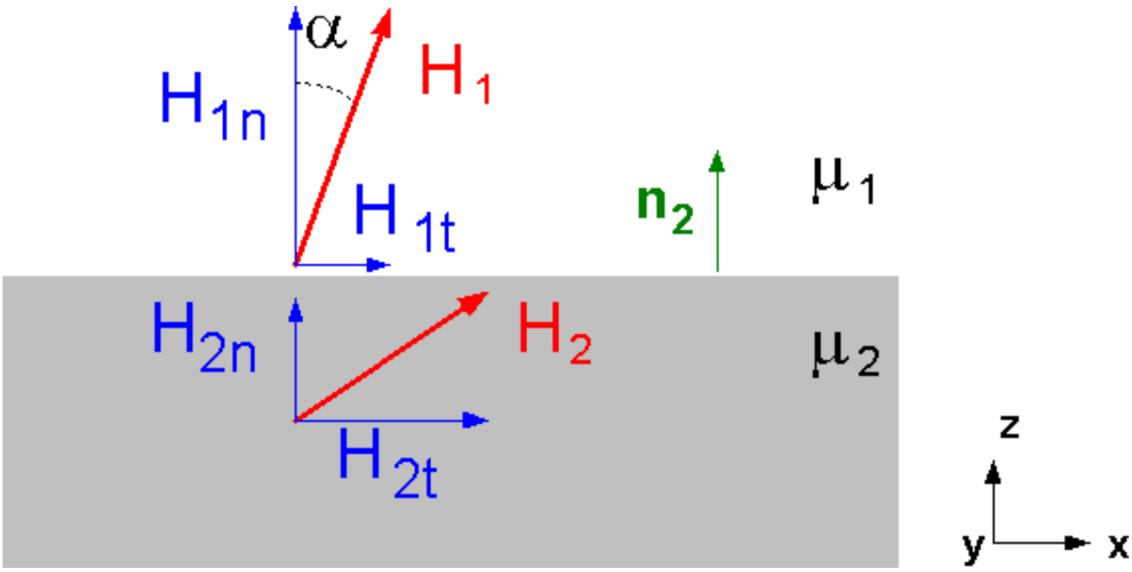
Thus

$$H_{2n} - H_{2n} = J_s$$

Exercise 5.12 (Fundamentals of electromagnetics…)

Determine the angle between $\mathbf{H}_1$ and $\mathrm{n}_2 = \hat{\mathrm{z}}1$

if $\mathrm{H}_2 = \hat{\mathrm{x}}3 + \hat{\mathrm{z}}2,\ \mu_1 = 2,\ \ \mu_2 = 8,\ J_s = 0$



Solution

$$\mu_1 \cdot \mathrm{H}_{1n} = \mu_2 \cdot \mathrm{H}_{2n}$$
$$\mathrm{H}_{2t} - \mathrm{H}_{1t} = J_s$$

Thus normal component of H_1 is

$$\mathrm{H}_{1n} = \frac{\mu_2 \cdot \mathrm{H}_{2n}}{\mu_1} = \frac{8 \cdot \hat{z}2}{2} = \hat{z}8$$

and tangential component of H_1 is

$$H_{2t} = H_{1t}$$
$$H_{1t} = 3$$

Thus

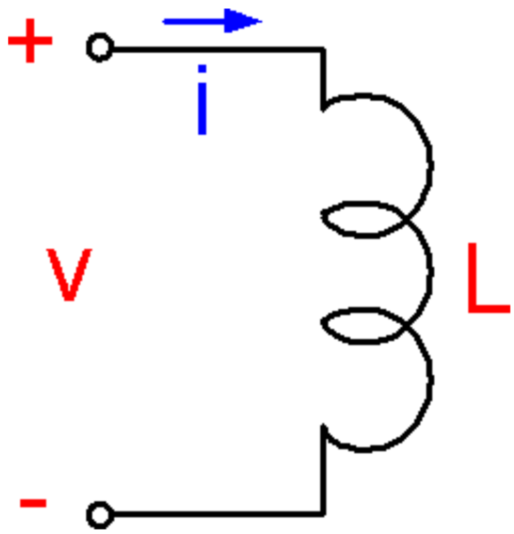
$$tg\alpha = \frac{H_{1t}}{H_{1n}} = \frac{3}{8} = 0.375$$

That gives

$$\alpha = arctg(0.375) = 20.55^{\circ}$$

Storage of Energy in a Magnetic Field

Consider first an air-cored coil characterized by its inductance *L*, and with current brought from zero to a value *i* in time *t* by application of a voltage *v*.



From the definition of inductance

$$v = L\frac{di}{dt}$$

and the instantaneous power is

$$p = vi$$

The energy input over a period of time is:

$$W = \int_0^t p \cdot dt$$

Thus

$$W = \int_0^t vi \cdot dt = \int_0^t L\frac{di}{dt} i \cdot dt = \int_0^t Li \cdot di = L\int_0^i i \cdot di = \frac{1}{2}Li^2$$

Just to remain you about the integral:

$$\int_a^b x \cdot dx = \left[\frac{x^2}{2}\right]_a^b + C = \frac{b^2}{2} - \frac{a^2}{2} + C$$

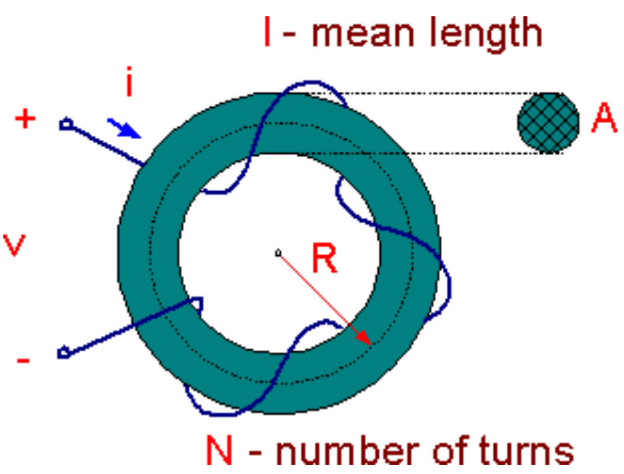
So that the energy stored in an inductor is

$$\frac{1}{2}Li^2$$

which compares with the energy stored in a capacitor

$$-\frac{1}{2}Cv^2$$

Now consider a toroidal coil as shown which is initially unmagnetized but in which v,i, and f are steadily built up over a period of time.



The electrical energy input is:

$$W = \int_0^t vi \cdot dt$$

But from Faraday's law:

$$v = N\frac{d\phi}{dt}$$

So that:

$$W = \int_0^t vi \cdot dt = \int_0^t N\frac{d\phi}{dt} i \cdot dt = N\int_0^t \frac{d\phi}{dt} i \cdot dt = N\int_0^\phi i \cdot d\phi$$

But by Ampere's law the total current linking the closed path around the center of the toroidal coil is *Ni* , therefore

$$Ni = Hl$$

Also

$$d\phi = A \cdot dB$$

So that

$$\begin{cases} Ni = Hl \\ d\phi = A \cdot dB \end{cases}$$

and

$$Ni \cdot d\phi = Hl \cdot A \cdot dB$$

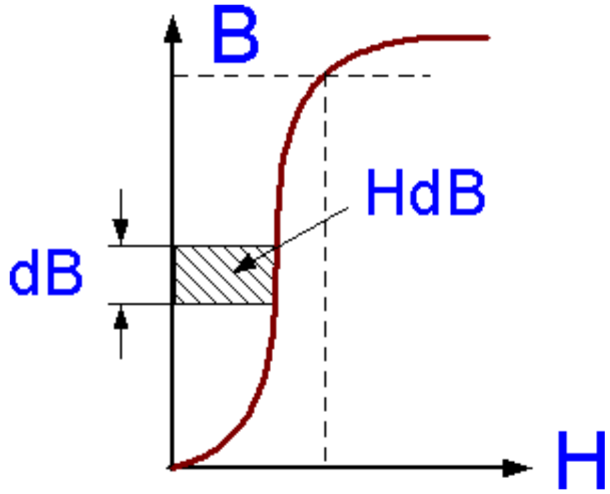
which is also

$$Ni \cdot d\phi = H \cdot dB \cdot (l \cdot A)$$

Thus the energy density is

$$W_d = \frac{W}{l \cdot A} = \int_0^B H \cdot dB$$

Which can be interpreted as the area between the magnetization curve and the B axis.



If the permeability is constant

$$B = \mu H$$

and the integral can be done

$$W_d = \int_0^B H \cdot dB = \int_0^B \frac{B}{\mu} \cdot dB = \frac{B^2}{2\mu} \qquad (J/m^3)$$

or

$$W_d = \frac{\mu H^2}{2}$$

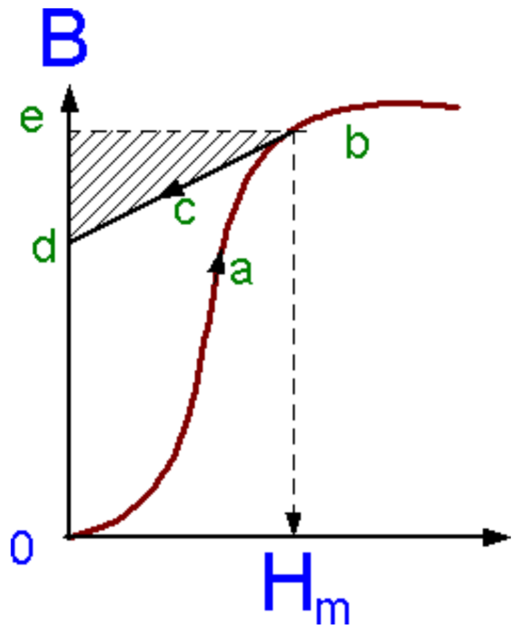
Physically energy is stored throughout the magnetic field with energy density

$$\frac{B^2}{2\mu}$$

Note that for given **B** lower value of m corresponds to higher energy storage.

Hysteresis Loss in a Ferromagnetic Core

Suppose the toroidal core of the previous example is of ferromagnetic material and is magnetized from zero along the curve OAB.



When $H=H_m$ the corresponding energy input is equal to the area ***oabe*** (J/m^3). Now let **H** to be reduced to zero, such that the magnetization curve follows ***bcd***. Than the energy represented by ***bcdeb*** is returned to the source. Thus in the process there is a **net energy loss** given by the area ***0abcd***.
If the magnetization is in a cyclical state the energy loss in the core (at very low frequencies) is the **area of the hysteresis loop** for each cycle of magnetization. This is the **Hysteresis Loss**.

In this cyclical process electrical energy is converted to heat as the domain walls are swept though the material. Low hysteresis loss materials (eg commercial silicon sheet steels) have narrow hysteresis loops giving that low loss. For these materials the systeresis loss per cycle has an empirical form determined by Steinmetz.

$$W_h = k_h \cdot (B_{\mathbf{max}})^n$$

where k_h and n must be empirically determined.

n is called the Steinmetz coefficient and its typically in the range 1.6-2.0.