

2 Mathematical Physics. I.

Matrix → Square or rectangular array of nos. or figs
obeying certain laws.

$$\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} \quad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad \begin{bmatrix} 1-i & 2+i \\ 4+2i & x+iy \\ 2+i & 3-2i \end{bmatrix}$$

The individual nos or figs are called the elements of the matrix.

if - The matrix has m rows and n columns is

A matrix with m rows and n columns ($m \times n$ matrix)
 called matrix of order $m \times n$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_m \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

$$A = [a_{ij}]_{m \times n} \rightarrow i \rightarrow \text{denotes row} \\ j \rightarrow \text{column}$$

$$1 \leq i \leq m \quad \text{and} \quad 1 \leq j \leq n$$

Algebraic operations of matrices.

1. Equality 2 matrices A [$= a_{ij}$] and B [$= b_{ij}$]

are said to be equal ($A = B$) if they have the same order $m \times n$ and each element of the first matrix is equal to the corresponding element of the other.

(i.e.) $a_{ij} = b_{ij}$ for all values of i & j .
 (i.e.) $(i=1, 2, \dots, m; j=1, 2, \dots, n)$

2. Addition & Subtraction

→ can be carried out if the orders of the matrix A & B are the same. The ~~same~~ if the 2 matrices $A = [a_{ij}]$ and $B = [b_{ij}]$, with order $m \times n$.

Then, the sum is $C = \{c_{ij}\}$ of order $m \times n$.

(2)

Matrix addition obeys the following properties:

1) the commutative law.

$$A + B = B + A \quad (\text{or}) \quad a_{ij} + b_{ij} = b_{ij} + a_{ij}$$

2) the associative law

$$A + (B + C) = (A + B) + C \quad (\text{or})$$

$$a_{ij} + (b_{ij} + c_{ij}) = (a_{ij} + b_{ij}) + c_{ij}$$

3). distributive law

$$\lambda(A + B) = \lambda A + \lambda B \quad (\text{or})$$

$$\lambda(a_{ij} + b_{ij}) = \lambda a_{ij} + \lambda b_{ij}$$

C. matrix
multiplication
properties

3. multiplication by scalar.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ then

Scalar $\lambda A = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \end{bmatrix}$

4. multiplication. Two matrices $A = [a_{ij}]$

$B = [b_{ij}]$ can be multiplied in the order AB only when the no. of columns in A is equal to no. of rows in B .

If A is of order $(m \times h)$ and B is of order $h \times n$, then $C = AB$ is of order $m \times n$. The elements are

$$c_{ij} = \sum_{k=1}^h a_{ik} b_{kj}$$

$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + a_{i3} b_{3j}$$

Properties of matrix multiplication

(3)

1) In the product AB or the matrix A & B , B is premultiplied by A and A is

post multiplied by B . In the product AB , A is called the prefactor and B the post-factor.

2) Matrix multiplication is not commutative.

i.e. $AB \neq BA$

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad BA = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\therefore AB \neq BA.$$

3) Associative law.

$$A(BC) = (AB)C$$

Let $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$ be 3 matrices of order $m \times n$, $n \times p$, $p \times q$ now AB is a $m \times p$ matrix.

4) Distributive law or multiplication

$$A(B+C) = \cancel{A}B + A\cancel{C}$$

Given $A = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$ + $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$

Find AB & BA & check whether they are equal.

Square matrix \rightarrow matrix having same no. of rows & columns is called a square matrix.

(eg) $\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \rightarrow a_{11}, a_{22}, a_{33}$

are called diagonal elements.

The sum of diagonal elements of a square matrix is called trace of the matrix.

Diagonal matrix. -

If all the elements of a square matrix are zero except those in the diagonal then it is called a diagonal matrix.

(eg) $\begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ 0 & 0 & a_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & a_{nn} \end{bmatrix} \rightarrow \text{diagonal matrix of order } n.$

(i) $n \times n$

Scalar matrix.

A diagonal matrix in which all the diagonal elements are equal, say λ , is called a scalar matrix.

$$\begin{bmatrix} \lambda & 0 & 0 & \dots & 0 \\ 0 & \lambda & 0 & \dots & 0 \\ 0 & 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda \end{bmatrix} \rightarrow n \times n \rightarrow \text{scalar matrix.}$$

Identity or unit matrix.

A scalar matrix in which the diagonal element is unity is called identity or unit matrix.

$$a_{ij} = \begin{cases} 1 & \text{for } i=j \\ 0 & \text{for } i \neq j \end{cases}$$

(5)

The unit matrix is denoted by I_n .

For a unit matrix I ,

$$IA = A \quad \text{and} \quad BI = B$$

Row & column matrices.

A matrix containing only one row or column is called a vector.

A matrix with only one row is called a row matrix (order $\rightarrow 1 \times n$) or a row vector.

$$[x]_{1 \times n} = [a_{11} \ a_{12} \ a_{13} \ \dots \ a_{1n}]$$

A matrix with only one column is called a column matrix or a column vector.

$$[x]_{m \times 1} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix} \quad (\text{order} \rightarrow m \times 1)$$

Null matrix or Zero matrix.

If all the elements a_{ij} in $m \times n$ matrix are zero, then the matrix is called null matrix or zero matrix. a_{ij} order $m \times n$.

Denoted by 0_{mn} or simply 0 .

$$(g) 0_{3 \times 4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow 3 \times 4 \text{ null matrix.}$$

For null matrix 0 , $0A = 0$, $A0 = 0$.

Upper triangular matrix.

A square matrix whose elements are $a_{ij} = 0$ for $i > j$ is called U.T. matrix.

(6)

$$\left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \hline 0 & 0 & 0 & \dots & a_{nn} \end{array} \right]$$

Upper triangular matrix.

Lower triangular matrix.

A square matrix with elements a_{ij} .If $a_{ij} = 0$ for $i < j$ then the matrix is L. t. matrix.

$$\left[\begin{array}{ccccc} a_{11} & 0 & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & 0 & \dots & 0 \\ a_{31} & a_{32} & a_{33} & 0 & \dots & 0 \\ \hline a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} & \end{array} \right]$$

Transpose.

By interchanging the rows & columns of an $m \times n$ matrix the transpose of a matrix can be obtained (order $n \times m$).It is denoted by A' or \tilde{A} or A^T .

If $A = \left[\begin{array}{ccccc} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{array} \right]$

then

$$A^T = \left[\begin{array}{ccccc} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & a_3 & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{array} \right]$$

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Some properties of transpose of a matrix.

(7)

Let $\overline{A^T}$ & $\overline{B^T}$ are the transposes of A & B .

a) $(\overline{A^T})^T = A$

b) $(\lambda A)^T = \lambda A^T$, ($\lambda \rightarrow \text{scalar}$)

c) $(A+B)^T = A^T + B^T$

d) $(AB)^T = B^T A^T$ \rightarrow reversal law of transpose

Conjugate of a matrix.

If a matrix have complex nos. then the conjugate matrix is obtained by replacing each element by its complex conjugate.

It is denoted by \overline{A} or A^*

If $A = [a_{ij}]$

then $A^* = [a_{ij}^*]$

where a_{ij}^* is the complex conjugate of a_{ij} .

If $A = \begin{bmatrix} 1+2i & 2-i \\ 3 & 5-2-3i \end{bmatrix}$ then

$A^* = \begin{bmatrix} 1-2i & 2+i \\ 3 & -5i-2+3i \end{bmatrix}$

Some properties of conjugate of a matrix.

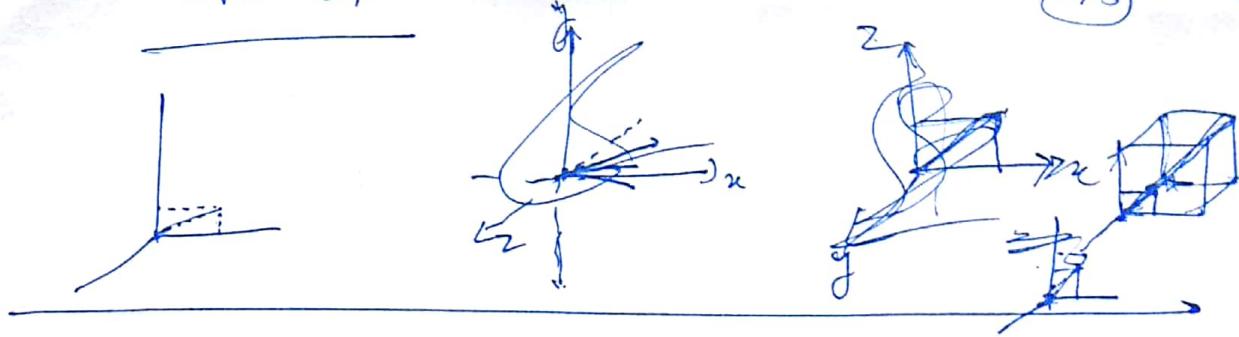
a) $(A^*)^* = A$

b) $(A+B)^* = A^* + B^*$

c) $(\lambda A)^* = \lambda^* A^*$ ($\lambda \rightarrow \text{any complex no.}$)

d) $(AB)^* = A^* B^*$

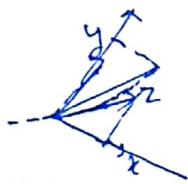
Vector Space



(8)

(43)

Consider 3 D Space



↪ consider a vector

described by 3 components

$a_1, a_2, a_3 \rightarrow$ components of vector
along $x, y, \& z$ axis.

The vector is represented by

$$a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$

Can be written as $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$$\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$

↖ R.H.S.

X →	\hat{i}
Y →	\hat{j}
Z →	\hat{k}

3-vector

Why $[a_1, a_2, a_3, \dots, a_n]$ is an n-vector.

An ordered set of n numbers (complex or real)

is called n-vector or n-dimensional vector.

and written as $\vec{a} = [a_1, a_2, a_3, \dots, a_n]$

or $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} = [a_1, a_2, a_3, \dots, a_n]^T$

The vector can be represented by
row or column matrix.

Then any mxn matrix \rightarrow m row vectors
& n column vectors.

Algebraic operations on vectors

$$(1) \vec{a} + \vec{b} = [a_1, a_2, a_3, \dots, a_n] + [b_1, b_2, b_3, \dots, b_n]$$

$$= [a_1 + b_1, a_2 + b_2, a_3 + b_3, \dots, a_n + b_n]$$

ii) If k is a scalar, then

(g)

$$k\bar{a} = k[a_1, a_2, \dots, a_n]$$

$$= [ka_1, ka_2, ka_3, \dots, ka_n]$$

(iii) $\bar{a} \cdot \bar{b} = a_1 * b_1 + a_2 * b_2 + \dots + a_n * b_n$

(iv) $\bar{a} \cdot \bar{a} = |\bar{a}|^2 = a_1 * a_1 + a_2 * a_2 + \dots + a_n * a_n$

$$= (a_1)^2 + (a_2)^2 + \dots + (a_n)^2$$

(v) A vector is a unit vector if $|\bar{a}| = 1$

(vi) The vectors \bar{a} & \bar{b} will be orthogonal if $\bar{a} \cdot \bar{b} = 0$ ($a \cdot b = ab \neq 0$)

(vii) The vector $\bar{0} = [0, 0, \dots, 0]^T$ is said to be a null vector so that $\bar{a} + \bar{0} = \bar{0} + \bar{a}$.

Linearly dependent and independent vectors.

A set of n -vectors $\bar{a}_1, \bar{a}_2, \bar{a}_3, \dots, \bar{a}_n$ are linearly dependent, if there exists a set of r non-zero scalars k_1, k_2, \dots, k_r such that $k_1 \bar{a}_1 + k_2 \bar{a}_2 + \dots + k_r \bar{a}_r = 0$.

(k 's $\neq 0$)

If $k_1 = k_2 = \dots = k_r = 0$, then

$\bar{a}_1, \bar{a}_2, \bar{a}_3, \dots, \bar{a}_r$ said to be linearly independent.

Vector Space

(6)

Any set of n -vectors over a field \mathbb{F} (which is closed w.r.t. addition and scalar multiplication) is called a vector space over field \mathbb{F} .

(magnetic field \rightarrow vectors can be defined according to magnetic lines of flux).

The collection of these vectors \rightarrow

(closed field \rightarrow no change in the magnitude & direction of the vectors due to interaction with other field
 \rightarrow zero)

The vector space \rightarrow denoted by $V_n(\mathbb{F})$ or V_n

If $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ are set of n vectors then the set of all linear combinations $k_1 \vec{a}_1 + k_2 \vec{a}_2 + \dots + k_n \vec{a}_n$ is also a vector space over \mathbb{F} , (k_1, k_2, \dots scalars)

Dimension of vector space:

\hookrightarrow maximum no. of linearly independent vectors in the space.

No. of directions can be defined in the space w.r.t. the ~~fixed~~ magnetic lines of force.

Affine A set of n linearly independent vectors in n -dimensional vector space V_n is called the basis for the vector space.

Basis is not unique \rightarrow no. of possible bases exists.

Orthonormal sets.

(11)

If the scalar product of two (non-null) vectors is zero, then the vectors are said to be orthogonal. So if $\vec{u} + \vec{v}$ are 2 vectors they are ~~orthogonal~~ \vec{u} \vec{v} ~~orthogonal if and only if~~ $(\vec{u}, \vec{v}) = 0$

If the scalar product of a vector with itself is unity, then the vector is said to be normalized to unity. i.e., if $\vec{u} \cdot \vec{u} = 1$.

A set of vectors which is orthogonal to all the remaining vectors of the set is called an orthogonal set.

If each of the vectors of the set is further normalized to unity, then it is called an orthonormal set.

If \vec{u} is any vector, then $\frac{\vec{u}}{\|\vec{u}\|}$ is the normalized vector.

A set of n vectors \vec{u}_i ($i=1, 2, 3, \dots, n$) is an orthonormal set if and only if

$$(\vec{u}_i, \vec{u}_j) = \delta_{ij} \quad (i, j = 1, 2, \dots, n)$$

where $\delta_{ij} \begin{cases} = 1 & \text{if } i=j \\ = 0 & \text{if } i \neq j \end{cases}$

$$\boxed{\begin{array}{l} (\vec{u}_1, \vec{u}_1) = 1 \\ (\vec{u}_1, \vec{u}_2) = 0 \\ \vdots \end{array}}$$

Schmidt's or the normalizing algorithm

(12)

↓
linearly independent to get a set of orthonormal
vectors

Let $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ be a set of linearly independent vectors (not necessarily orthogonal)

we have to get another set of
(orthogonal vectors) $\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_n$

from the original set.

① choose $\vec{v}_1 = \vec{u}_1$

② Let $\vec{v}_2 = \vec{u}_2 - a_{21} \vec{v}_1$

Take scalar product of \vec{v}_1 & \vec{v}_2 & equate

$$(\vec{v}_1, \vec{v}_2) = (\vec{v}_1, \vec{u}_2 - a_{21} \vec{v}_1) \xrightarrow{\text{to } 0}$$

\hookrightarrow constant
 \hookrightarrow determined from the condition that \vec{v}_2 is orthogonal to \vec{v}_1 , i.e. $(\vec{v}_1, \vec{v}_2) = 0$

$$(\vec{v}_1, \vec{u}_2) - a_{21} (\vec{v}_1, \vec{v}_1) = 0$$

$$a_{21} = \frac{(\vec{v}_1, \vec{u}_2)}{(\vec{v}_1, \vec{v}_1)}$$

Now we have \vec{v}_1 & \vec{v}_2 orthogonal.

③ Let $\vec{v}_3 = \vec{u}_3 - a_{31} \vec{v}_1 - a_{32} \vec{v}_2$

\downarrow
constants
determined using the condition \vec{v}_3 is orthogonal to \vec{v}_1 & \vec{v}_2

or orthogonal

$$(\vec{v}_1, \vec{v}_3) = 0 = (\vec{v}_1, \vec{u}_3 - a_{31} \vec{v}_1 - a_{32} \vec{v}_2) \xrightarrow{\text{to } 0}$$

$$0 = (\vec{v}_1, \vec{u}_3) - a_{31} (\vec{v}_1, \vec{v}_1) - a_{32} (\vec{v}_1, \vec{v}_2) \xrightarrow{\text{to } 0}$$

$$0 = (\vec{v}_1, \vec{u}_3) - a_{31} (\vec{v}_1, \vec{v}_1) \xrightarrow{\text{to } 0} a_{31} = \frac{(\vec{v}_1, \vec{u}_3)}{(\vec{v}_1, \vec{v}_1)}$$

(13)

$$\text{Thus } (\vec{v}_1, \vec{v}_3) = 0$$

$$\text{now } (\vec{v}_2, \vec{v}_3) = (\vec{v}_2, v_3 - q_{32} \vec{v}_2 - q_{31} \vec{v}_1)$$

$$0 = (\vec{v}_2, \vec{v}_3) - q_{32}(\vec{v}_2, \vec{v}_2) - q_{31}(\vec{v}_2, \vec{v}_1)$$

$$(\vec{v}_2, \vec{v}_3) = q_{32}(\vec{v}_2, \vec{v}_2)$$

$$q_{32} = \frac{(\vec{v}_2, \vec{v}_3)}{(\vec{v}_2, \vec{v}_2)}$$

now $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are orthogonal.

Since $\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_n$ which are mutually

orthogonal. (The appropriate constants ~~have also been~~ can also be derived)

In general, the i^{th} element

v_i is written as,

$$v_i = u_i - a_{i1} v_1 - a_{i2} v_2 - \dots - a_{i(i-1)} v_{i-1}$$

$$\text{and } a_{ij} = \frac{(\vec{v}_j, \vec{u}_i)}{(\vec{v}_j, \vec{v}_j)}$$

(The coefficients $a_{ij}, a_{i2}, \dots, a_{i(i-1)}$)

↳ derived on condition v_i is orthogonal to all the preceding elements $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}$.

All vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ can be normalized to obtain an orthonormal set $\{\vec{x}_i\}$.

$$(i.) \vec{x}_i = \frac{\vec{v}_i}{(\vec{v}_i, \vec{v}_i)^{1/2}} = \frac{\vec{v}_i}{\sqrt{\sum_{i=1}^n v_i^2}}$$

Linear transformations.

(14)

Let $\vec{X} = [x_1, x_2, \dots, x_n]$ and
 $\vec{Y} = [y_1, y_2, \dots, y_n]$ be two vectors

in vector space $V_n(F)$

Let the ~~separate~~ components of \vec{X} and \vec{Y}
be related as follows.

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = y_1$$

$$a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n = y_2$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{n1} x_1 + a_{n2} x_2 + \dots + a_{nn} x_n = y_n$$

The sep. eqns can be represented
using matrix notation.

$$AX = Y \quad \text{--- (1)}$$

$$\text{where } A = [a_{ij}]_{n \times n}$$

Equation (1) corresponds to a transformation
because it involves a transformation y ,
vector \vec{X} into vector \vec{Y}

A is called the matrix of transformation.

If the transformation in eqn (1)
transforms \vec{X}_1 into \vec{Y}_1
and \vec{X}_2 into \vec{Y}_2 , then

- 1) it transforms $k\vec{X}_1$ into $k\vec{Y}_1$ for every scalar k
 - 2) it transforms $k_1\vec{X}_1 + k_2\vec{X}_2$ into $k_1\vec{Y}_1 + k_2\vec{Y}_2$
- $\vec{X} + \vec{Y}$ are related linearly & hence the transformation is called linear transformation

(15)

If $|A| \neq 0$ then the transformation
is called non-singular

If $|A| = 0$, ... singular

The rank of the transformation matrix A is
said to be the rank of the transformation

If $A \neq 0$, then

$$AX = y$$

$$A^{-1}AX = A^{-1}y$$

$$X = A^{-1}y$$

\rightarrow There is one to one
correspondence between
 \vec{x} and \vec{y}

Consider $A\vec{x} = \vec{y}$ (2)

and $B\vec{y} = \vec{z}$ (3)

then $\vec{z} = B(A\vec{x})$ (3)

It is called the product of
transformations. It is the resultant of
transformations (2) & (3).

Eigen values and Eigen vectors

Consider the transformation

$$A\vec{x} = \lambda\vec{x} \quad (1)$$

\downarrow
Scalar

Square matrix of order n .

(1) can be re-written as

$$(Ax - \lambda x) = 0$$

$$(A - \lambda I)x = 0$$

$$(A - \lambda I)x = 0 \quad (2)$$

\downarrow
unit matrix.

The value λ for which ① or ② (16) have non-zero solution ($X \neq 0$) is called eigen value, or characteristic root or latent root of the matrix A .

The Corresponding non zero Solution X is called eigen vector, or characteristic vector or latent vector of the value λ .

The matrix $(A - \lambda I)$ is called the characteristic matrix of A .

The determinant $\phi(\lambda) = |A - \lambda I|$ is called the characteristic polynomial of A .

Eqn ① or ② will be ^{now} found only if

$$\phi(\lambda) = |A - \lambda I| = 0 \quad \text{--- (3)}$$

$$(A - \lambda I)X = 0$$

If 0, then
it's trivial
solution

Hence
 $(A - \lambda I)$
Should be 0.

$$\phi(\lambda) = a_0 + a_1 \lambda + a_2 \lambda^2 +$$

$$\dots \dots \dots a_n \lambda^n = 0$$

↳ characteristic polynomial.

③ is called
Secular equation
or characteristic eqn.

Eg

Find the eigen values and normalized eigen vectors of the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(17)

Solutions

$$\text{Let } A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix} \end{aligned}$$

The characteristic equation is

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} =$$

$$(1-\lambda) [(1-\lambda)^2 - 1] = 0$$

$$(1-\lambda) (\lambda^2 + \lambda^2 - 2\lambda + 1) = 0$$

$$(1-\lambda) (\lambda^2 - 2\lambda) = 0$$

$$(1-\lambda) \lambda (\lambda - 2) = 0$$

i. $\lambda (1-\lambda)(\lambda-2) = 0$

$$\lambda = 0 \quad \therefore \quad \therefore$$

$$(1-\lambda) = 0 \rightarrow \lambda = 1$$

$$(\lambda - 2) = 0 \rightarrow \lambda = 2$$

$$\lambda = 0, 1, 2$$

The eigen values of the matrix are

$$\lambda = 0, 1, 2$$

The eigen value $\lambda = 0$ in $(A - \lambda I)x = 0$

For $\lambda = 0$, $\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} - 0I \right) \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

(18)

$$\lambda_1 = 0$$

$$x_2 + x_3 = 0, \quad x_2 = -x_3$$

~~∴~~

1 term $x_1 = 0$
 $\lambda = -k_3$

The eigen vector corresponding to $\lambda = 0$

$$\vec{X}_1 = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \{ x_1, x_2, x_3 \} = \{ 0, k, -k \}$$

If the eigen vector is normalized to unity,
then $(\vec{X}_1) = 1$

$$\therefore \sqrt{0^2 + k^2 + (-k)^2} = 1$$

$$\sqrt{2k^2} = 1$$

$$k\sqrt{2} = 1$$

$$k = \frac{1}{\sqrt{2}}$$

$$\therefore \vec{X}_1 = \left\{ 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\}$$

For $\lambda = 1$,

$$(A - \lambda I) \vec{X} = 0$$

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \vec{X} = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \vec{X} = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = 0$$

$$x_2 = 0$$

$$x_3 = 0$$

(19)

$$\therefore \vec{X}_2 = \{x_1, x_2, x_3\} = \{0, 0, 0\}$$

For $\lambda=2$

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \vec{X} = 0$$

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \vec{X} = 0$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + 0x_2 + 0x_3 = 0 \rightarrow -x_1 = 0, x_1 = 0$$

$$0x_1 - x_2 + x_3 = 0 \rightarrow x_3 = x_2$$

$$0x_1 + x_2 - x_3 = 0 \rightarrow x_3 = x_2$$

\therefore The eigen vector $\vec{X}_3 = \{x_1, x_2, x_3\}$
 $= \{0, k, k\}$

The normalized eigen vector \vec{X}_3

$$\text{i} \quad \sqrt{\omega^2 + k^2 + k^2} = 1$$

$$\sqrt{2k^2} = 1$$

$$\sqrt{2}k = 1, k = \frac{1}{\sqrt{2}}$$

$$\therefore \vec{X}_3 = \{0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\}$$

Find the eigen values and normalized eigen vectors of the following matrix.

$$\begin{bmatrix} 3 & 4 & 7 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

Find the eigen values and eigen vectors

Q- $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

Eigen values:

The characteristic equation of A is $\det \begin{bmatrix} \cos \theta - \lambda & -\sin \theta \\ \sin \theta & \cos \theta - \lambda \end{bmatrix} = 0$

$$\begin{vmatrix} \cos \theta - \lambda & -\sin \theta \\ \sin \theta & \cos \theta - \lambda \end{vmatrix} = 0$$

$$(\cos \theta - \lambda)^2 + \sin^2 \theta = 0$$

$$\cos^2 \theta + \lambda^2 - 2\lambda \cos \theta + \sin^2 \theta = 0$$

$$1 + \lambda^2 - 2\lambda \cos \theta = 0, \quad \lambda^2 - 2\lambda \cos \theta + 1 = 0$$

$$\lambda = \frac{2 \cos \theta \pm \sqrt{4 \cos^2 \theta - 4}}{2} = \cancel{\frac{2 \cos \theta + 2 \cos \theta}{2}}$$

$$= \frac{2 \cos \theta \pm 2 \sqrt{(\cos^2 \theta - 1)}}{2} = \cancel{2 \cos \theta} \pm$$

$$\therefore \cos \theta \pm i \sin \theta = e^{\pm i \theta}$$

$$\therefore \lambda_1 = e^{+i\theta}, \quad \lambda_2 = e^{-i\theta}$$

$$\left. \begin{aligned} \cos^2 \theta + \sin^2 \theta &= 1 \\ \cos^2 \theta - 1 &= -\sin^2 \theta \\ -(1) \sin^2 \theta &= -\sin^2 \theta \\ \sin^2 \theta &= 0 \\ i^2 &= -1 \\ i \times i &= -1 \\ i &= -1 \end{aligned} \right\}$$

Eigen vectors

Q11

$$\begin{bmatrix} \cos\theta - 1 & -\sin\theta \\ \sin\theta & \cos\theta - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(\cos\theta - 1) x_1 - \sin\theta x_2 = 0$$

$$\sin\theta x_1 + (\cos\theta - 1) x_2 = 0$$

$$\text{Let } \lambda = \lambda_1 = e^{i\theta} = \cos\theta + i\sin\theta$$

$$(\cos\theta - 1) x_1 = \sin\theta x_2$$

$$\sin\theta x_1 + (\cos\theta - 1) x_2 = 0$$

$$(\cos\theta - \cos\theta - i\sin\theta) x_1 = \sin\theta x_2$$

~~$x_2 = ix_1 = x_2$~~

∴ The eigen vector is

$$X_1 = \{x_1, x_2\} = \{x_1, -ix_1\} = x_1 \{1, -i\}$$

$$\text{Now } \lambda = \lambda_2 = e^{-i\theta} = \cos\theta - i\sin\theta$$

$$(\cos\theta - 1) x_1 - \sin\theta x_2 = 0$$

$$\sin\theta x_1 + (\cos\theta - 1) x_2 = 0$$

$$(\cos\theta - \cos\theta + i\sin\theta) x_1 - \sin\theta x_2 = 0$$

$$\sin\theta x_1 + (\cos\theta - \cos\theta + i\sin\theta) x_2 = 0$$

$$i\sin\theta x_1 = \sin\theta x_2$$

$$ix_1 = x_2$$

$$i\sin\theta x_1 = -i\sin\theta x_2, x_1 = -ix_2$$

\therefore The eigen vector \vec{X}_2 is

$$\vec{X}_2 = \{x_1, x_2\} = \{\bar{x}_1, ix_2\} = x_1 \{1, i\}$$

or $\{1, -i\}$

(92)

~~Important~~ Theorems of Eigen values
and Eigen vectors.

(1) The eigen values of a hermitian matrix are real

$$A^T \rightarrow A \text{ is Hermitian}$$
$$(A^T)^* = A$$

Proof.

For a hermitian matrix $A^T = A$

(A^T is the transposed conjugate of A)

$$\text{Let } A \vec{x} = \lambda \vec{x} \quad \text{--- (1)}$$

$\vec{x} \rightarrow$ any eigen vector of A
corresponding to the eigen value λ .

Premultiply by \vec{x}^T ,

$$\vec{x}^T A \vec{x} = \lambda \vec{x}^T \vec{x}$$

$$\vec{x}^T A^T \vec{x} = \lambda^* \vec{x}^T \vec{x}$$

$$(\vec{x}^T A \vec{x})^T = (\lambda \vec{x}^T \vec{x})^T$$

$$\vec{x}^T A^T (\vec{x}^T)^T = \lambda^* \vec{x}^T (\vec{x}^T)^T$$

$$\vec{x}^T A^T \vec{x} = \lambda^* \vec{x}^T \vec{x}$$

$$\underbrace{\vec{x}^T A \vec{x}}_{A^T = A \text{ for Hermitian matrix}} = \lambda^* \vec{x}^T \vec{x}$$

$$\lambda^* \vec{x}^T \vec{x} = \lambda^* \vec{x}^T \vec{x}$$

$$(\lambda - \lambda^*) \vec{x}^T \vec{x} = 0$$

X is an eigen vector, $X \neq 0$, $\therefore X^T X \neq 0$

$$\therefore (\lambda - \lambda^*) = 0$$

(Q3)

$$\lambda = \lambda^*$$

The conjugate of λ is equal to itself.

This is possible only if λ is real.

Thus the Eigen values of a Hermitian matrix are real.

(Q2) The Eigen values of a real Symmetric matrix are real.

Proof.

For a real Symmetric matrix,

$$A^* = A \rightarrow (1) \quad A^T = A \rightarrow (2)$$

Take complex conjugate of

$$(A^T)^* = A^*$$

$$= A$$

$$A^+ = A$$

\therefore The real Symmetric matrix is a Hermitian matrix. The Eigen values of a Hermitian matrix are all real. Hence the eigen values of a real Symmetric matrix are real.

(3) The eigen values of a skew-Hermitian matrix are either zero or purely imaginary

Proof.

If A is skew-Hermitian, then

$$A^+ = -A$$

Let X be an eigen vector corresponding to λ .

$$AX = \lambda X$$

$$(i^{\circ} A)X = (i\lambda)X$$

$$(i^{\circ} A)^T = i^{\circ} A^T = i^{\circ} \lambda^T$$

$$(i^{\circ} A)^T = -i^{\circ} (-A) \quad (A^T = -A)$$

$\therefore i^{\circ} A$ is Hermitian matrix

$(i^{\circ} A)X = (i\lambda)X \rightarrow$ implies $i^{\circ} \lambda$ is the eigen value of the hermitian matrix $i^{\circ} A$.

The eigen values of a hermitian matrix are real.

$\therefore i^{\circ} \lambda$ is real

$\therefore \lambda$ is 0 (or) imaginary.

④ The Eigen values of a real skew-symmetric matrix are either 0 or imaginary.

Proof.

For a real skew-symmetric matrix A

$$A^* = A \rightarrow ①$$

$$A^T = -A \rightarrow ②$$

Take complex conjugate of ②

$$(A^T)^* = -\cancel{(A^*)} \rightarrow A$$

$$A^+ = -A$$

\therefore Matrix A is skew-Hermitian.

The eigen values of a skew-Hermitian matrix are 0 or imaginary.

\therefore The eigen values of a real skew-symmetric matrix are 0 or imaginary.

⑤ The Eigen values λ of a unitary matrix are of unit modulus. (25)

Proof.

For a unitary matrix,

$$A^T A = I \quad ①$$

Let x be an eigen vector of matrix A corresponding to the eigen value λ .

$$Ax = \lambda x \quad ②$$

$$(Ax)^T = (\lambda x)^T$$

$$x^T A^T = \lambda^* x^T \quad ③$$

Post multiply ② by ③

$$(x^T A^T)(Ax) = (\lambda^* x^T)(\lambda x)$$

$$x^T (A^T A)x = \lambda \lambda^* x^T x$$

$$x^T I x = \lambda \lambda^* x^T x$$

$$x^T x = \lambda \lambda^* x^T x$$

$$x^T x (1 - \lambda \lambda^*) = 0$$

$x \neq 0$, $\therefore x^T x \neq 0$, Hence

$$(1 - \lambda \lambda^*) = 0$$

$$\lambda \lambda^* = |\lambda|^2 = 1$$

$$|\lambda| = 1$$

\therefore The eigen values of a unit matrix are of unit modulus.

⑥ The Eigen values of an orthogonal matrix are unimodular.

Proof

For an orthogonal real matrix A ,

$$A^* = A \quad ①$$

$$A^T A = I \quad ②$$

Take complex conjugate of ②

(26)

$$(A^T A)^* = I^*$$

$$(A^T)^* A^* = I^*$$

$$(A^T)^* A^* = I \quad (I^* = I)$$

$$A^+ A^* = I$$

$$A^+ A = I \quad (A^* = A)$$

\therefore Matrix A is unitary

But the Eigen values of a unitary matrix are ~~real~~ unimodular. Hence the eigen values of an ~~is~~ orthogonal matrix are unimodular.

(7) The eigen values of a diagonal matrix are precisely the elements in the diagonal.

Proof.

$$\text{Let } A = \text{diag}[a_{11}, a_{22}, \dots, a_{nn}]$$

$$(A - \lambda I) = \text{diag}[a_{11} - \lambda, a_{22} - \lambda, \dots, a_{nn} - \lambda]$$

The characteristic polynomial

$$(A - \lambda I) = 0 = (a_{11} - \lambda)(a_{22} - \lambda) \dots (a_{nn} - \lambda)$$

$$\therefore (a_{11} - \lambda)(a_{22} - \lambda) \dots (a_{nn} - \lambda) = 0$$

$$\therefore (a_{11} - \lambda) = 0, \lambda = a_{11}$$

$$(a_{22} - \lambda) = 0, \lambda = a_{22}$$

$$\dots (a_{nn} - \lambda) = 0, \lambda = a_{nn}$$

$a_{11}, a_{22}, \dots, a_{nn}$ are the diagonal elements of A.

\therefore The eigen values of a diagonal matrix are the diagonal elements.

$$\begin{array}{c} \boxed{\lambda} \\ \boxed{a_{11}} \boxed{0} \boxed{0} \\ \boxed{0} \boxed{a_{22}} \boxed{0} \\ \boxed{0} \boxed{0} \boxed{a_{33}} \\ \vdots \vdots \vdots \\ 0 \ 0 \ \dots a_{nn} \end{array}$$

$$(a_{11} - \lambda)(a_{22} - \lambda) \dots (a_{nn} - \lambda)$$

(8) Any two Eigen vectors corresponding to two distinct eigen values of Hermitian matrix are orthogonal. (Q7)

Proof

Let x_1 & x_2 be two given eigen vectors corresponding to two distinct eigen values λ_1 & λ_2 of a Hermitian matrix A .

$$A^T = A \quad (\text{Hermitian}) \quad (1)$$

$$AX_1 = \lambda_1 x_1 \quad \rightarrow (2)$$

$$AX_2 = \lambda_2 x_2 \quad (3)$$

The eigen values of Hermitian matrix are real.
 $\therefore \lambda_1$ & λ_2 are real numbers.

$$\lambda_1 = \lambda_1^* \quad \lambda_2 = \lambda_2^*$$

From multiplying (2) ~~and~~ by x_2^T

$$x_2^T A x_1 = x_2^T \lambda_1 x_1 \quad (4)$$

Multiplying (3) by x_1^T

$$x_1^T A x_2 = \lambda_2 x_1^T x_2 \quad (5)$$

* Take transpose on both sides of (4)

$$(x_2^T A x_1)^T = (x_2^T \lambda_1 x_1)^T$$

$$x_1^T A^T (x_2^T)^T = \lambda_1^* x_1^T (x_2^T)^T$$

$$x_1^T A^T x_2 = \lambda_1^* x_1^T x_2 \quad (\lambda_1 = \lambda_1^*)$$

$$x_1^T A x_2 = \lambda_1 x_1^T x_2 \rightarrow (6)$$

Compare (5) & (6)

$$\lambda_2 x_1^T x_2 = \lambda_1 x_1^T x_2$$

$$(\lambda_1 - \lambda_2) x_1^T x_2 = 0$$

$$(\lambda_1 - \lambda_2) x_1 + x_2 = 0$$

(22)

λ_1 & λ_2 are the eigen values which are different. $\therefore (\lambda_1 - \lambda_2) \neq 0$ i.e. $\lambda_1 \neq \lambda_2$
 $\therefore x_1 + x_2 = 0$

$\therefore x_1$ & x_2 are orthogonal

⑨ Any two eigen vectors of a real symmetric matrix are orthogonal if the corresponding eigen values are different.

Proof For a real symmetric matrix A ,

$$A^* = A \quad ①$$

$$A^T = A \quad ②$$

Take transpose conjugate of ②

$$(A^T)^* = A^*$$

$$A^T = A^*$$

$$A^T = A$$

A is a Hermitian matrix

Hence a real, symmetric matrix is Hermitian

Theorem Any 2 e. vectors corresponding to two different eigen values of a hermitian matrix are orthogonal. Hence, the any 2 e. vectors of a real symmetric matrix are orthogonal if the e. values are different.

(b) Any 2 eigen vectors corresponding to two distinct eigen values of a unitary matrix are orthogonal. (2-25)

Proof

⑩ Any two Eigen vectors corresponding to two distinct Eigen values of a unitary matrix are orthogonal. (29)

If A is an unitary matrix then,

$$A^T A = I \rightarrow ①$$

Let x_1 and x_2 be two Eigen vectors for two distinct Eigen values λ_1 and λ_2 of a unitary matrix.

$$A x_1 = \lambda_1 x_1 \quad ②$$

$$A x_2 = \lambda_2 x_2 \quad ③$$

From ②

$$(A x_1)^T = (\lambda_1 x_1)^T$$

$$x_1^T A^T = x_1^T \lambda_1^T = \lambda_1^* x_1^T \quad ④$$

Now multiply ④ by ③

$$(x_1^T A^T) A x_2 = (\lambda_1^* x_1^T) \lambda_2 x_2$$

$$x_1^T (A^T A) x_2 = \lambda_1^* \lambda_2 x_1^T x_2$$

$$x_1^T I x_2 = \lambda_1^* \lambda_2 x_1^T x_2 \rightarrow \text{using } ①$$

$$x_1^T x_2 = \lambda_1^* \lambda_2 x_1^T x_2 \quad (x_2 = I x_2)$$

$$(1 - \lambda_1^* \lambda_2) x_1^T x_2 = 0 \rightarrow ⑤$$

But the Eigen values of a unitary matrix are of unit modulus. $\therefore \lambda_1^* \lambda_2 = 1$

$$(1 - \lambda_1^* \lambda_2) = (\lambda_1^* \lambda_1 - \lambda_1^* \lambda_2) = \lambda_1^* (\lambda_1 - \lambda_2) \neq 0$$

because λ_1 & λ_2 are distinct

$$\therefore x_1^T x_2 = 0$$

$\therefore x_1$ and x_2 are orthogonal.

Adjoint

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Square matrix

det A =

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

$$\begin{bmatrix} a_{22} \ a_3 \ \dots \ a_{2n} \\ a_{32} \ \dots \ a_{3n} \\ \vdots \\ a_{n2} \ \dots \ a_{nn} \end{bmatrix}$$

→ Square matrix

$$\text{adjoint} = \begin{bmatrix} a_{22} \ a_3 \ \dots \ a_{2n} \\ a_{32} \ \dots \ a_{3n} \\ \vdots \\ a_{n2} \ \dots \ a_{nn} \end{bmatrix}$$

adjoint = ~~[]~~

Let $A = [a_{ij}]$ n square matrix $B = [b_{ij}] \rightarrow (n-1) \times n$ square matrix b_{ij} is a co-factor of a_{ij} in the determinant $|A|$.The B is called the adjoint of A
denoted by $\text{adj } A$

$$\text{Property} \rightarrow (\text{adj } A)A = A(\text{adj } A) = |A| I_n$$

Adjoint of a matrix A is defined as the transpose of the matrix formed by the co-factors of the elements of $|A|$
 $\boxed{(n-1) \times n}$

Cayley-Hamilton theorem

(30)

Every square matrix satisfies its own characteristic equation, i.e., if A is a square matrix of order n the characteristic polynomial is

$$(A - \lambda I) = a_0 + a_1 \lambda + a_2 \lambda^2 + \dots + a_n \lambda^n$$

The matrix equation is

$$a_0 I + a_1 X + a_2 X^2 + \dots + a_n X^n$$

The matrix equation is satisfied by $X = A$.

Proof.

The characteristic polynomial of A is

$$(A - \lambda I) = 0 = a_0 + a_1 \lambda + a_2 \lambda^2 + \dots + a_n \lambda^n$$

The characteristic eqn is.

$$(A - \lambda I) = 0 = a_0 + a_1 \lambda + a_2 \lambda^2 + \dots + a_n \lambda^n = 0$$

The matrix equation is

$$(A - \lambda I) = 0 = a_0 I + a_1 X + a_2 X^2 + \dots + a_n X^n = 0$$

If this eqn is satisfied by A , then we have to show that

$$a_0 I + a_1 A + a_2 A^2 + \dots + a_n A^n = 0$$

$A \rightarrow$ order n (square matrix)

Let $B = \text{adj}(A - \lambda I) \rightarrow$ order $(n-1)$

Hence B ~~adjoint will have~~ a polynomial of degree $(n-1)$
 $\therefore B = \text{adj}(A - \lambda I) = B_0 + B_1 \lambda + B_2 \lambda^2 + \dots + B_{n-1} \lambda^{n-1}$
 we know, $(A - \lambda I) \text{ adj}(A - \lambda I) = (A - \lambda I) I$

$$(A - \lambda I) B = |A - \lambda I| I$$

$$(A - \lambda I) [B_0 + B_1 \lambda + B_2 \lambda^2 + \dots + B_{n-1} \lambda^{n-1}]$$

$$= (a_0 + a_1 \lambda + a_2 \lambda^2 + \dots + a_n \lambda^n) I$$

$$AB_0 + AB_1 \lambda + AB_2 \lambda^2 + \dots + AB_{n-1} \lambda^{n-1}$$

$$-B_0 \lambda I - B_1 \lambda^2 I - B_2 \lambda^3 I \dots - B_{n-1} \lambda^n I$$

$$= a_0 I + a_1 \lambda I + a_2 \lambda^2 I + \dots + a_{n-1} \lambda^{n-1} I$$

Compare the coefficients w.r.t. like powers of λ .

$$AB_0 = a_0 I$$

$$AB_1 - B_0 = a_1 I$$

$$AB_2 - B_1 = a_2 I$$

\vdots

$$AB_{n-1} - B_{n-2} = a_{n-1} I$$

$$-B_{n-1} = a_n I$$

Pre multiply these equations by $I, A, A^2, A^3, \dots, A^n$.

$$\cancel{IAB_0} + \cancel{A^2 B_1} - \cancel{AB_0} + \cancel{A^3 B_2} - \cancel{A^2 B_1} \\ \cancel{- A^{n-1} B_{n-1}} - A^n B_{n-1} = a_0 I + a_1 A + a_2 A^2 + \dots + a_n A^n$$

$$O = a_0 I + a_1 A + a_2 A^2 + \dots + a_n A^n$$

Find the characteristic equation of the following matrix and verify Cayley-Hamilton theorem. (32)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix}$$

Ans

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix}$$

$$\text{S.O.L} \quad A - \lambda I = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & 1-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda)(1-\lambda) - 4 - 2[2(1-\lambda) - 12] + 3[2 - 3(-1-\lambda)]$$

=

$$(1-\lambda)[(-1-\lambda)(1-\lambda) - 4] - 2[2(1-\lambda) - 12]$$

$$+ 3[2 - 3(-1-\lambda)]$$

$$= -\lambda^3 + \lambda^2 + 18\lambda + 30$$

$$\begin{array}{c} (-1)^3 + 1^2 + 1^2 + (-1) \\ (-1) + (4) + (2) + \\ (+6) + (9) + (9) + (-1) \\ \hline -1^3 + 1^2 + 3^2 + \\ + 141 \end{array}$$

$$\begin{array}{c} 1-\lambda \\ -1-\lambda \\ \hline \cancel{2} + \cancel{1} 2 \\ -1 \cancel{-1} \\ 1^2 - 1 \\ \hline -\lambda^3 + 1^2 + 1 - 1 \\ -1^3 + 1^2 + 3 - 5 \\ \hline -1^3 + 1^2 + 3 - 5 \end{array}$$

$$\begin{array}{c} 2 - 21 - 12 \\ -2 \\ \hline -4 + 41 + 24 \\ 3[2 + 3 + 31] \\ \hline 6 + 9 + 91 \end{array}$$

$$\begin{array}{c} 2 \\ 4 \\ \hline 4(4) - 24 \\ 6 - 9(-1-\lambda) \\ \hline 6 + 9 + 91 \end{array}$$

The characteristic equation of the matrix A is $-t^3 + t^2 + 18t + 30 = 0$ (33)

To verify C-H. theorem,
we show that-

$$\text{where } I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{bmatrix}$$

$$A^3 - A^2 A = \begin{bmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 62 & 39 & 68 \\ 48 & 21 & 78 \\ 62 & 24 & 62 \end{bmatrix}$$

$$\begin{aligned} & -A^3 + A^2 + 18A + 30I \\ &= -\begin{bmatrix} 62 & 39 & 68 \\ 48 & 21 & 78 \\ 62 & 24 & 62 \end{bmatrix} + \begin{bmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{bmatrix} \\ &+ 18 \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} + 30 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

(34)

Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

& verify C.H. Theorem.

Find the inverse of A .

$$(a) A^3 - 6A^2 + 9A - 4I = 0$$

$$4I = (A^3 - 6A^2 + 9A)$$

$$I = \frac{1}{4}(A^3 - 6A^2 + 9A)$$

Now multiply by A^{-1}

$$A^{-1} = \frac{1}{4}(A^2 - 6A + 9)$$

$\frac{-1}{4} \quad \frac{1}{4} \quad \rightarrow A^{-1}$ can be found.

Diagonalization of matrices.

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ are n distinct

Eigen values of a diagonalizable matrix A

and $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ be the corresponding

Eigen vector. \vec{x}_i is given by

$$\vec{x}_i = \begin{bmatrix} x_{1i} \\ x_{2i} \\ x_{3i} \\ \vdots \\ x_{ni} \end{bmatrix}$$

$$\vec{x}_i = \vec{x}_1 \text{ (say, having components } \begin{bmatrix} x_{1i} \\ x_{2i} \\ x_{3i} \\ \vdots \\ x_{ni} \end{bmatrix} \text{)} \quad \vec{x}_i = \vec{x}_2 \text{ (say, having components } \begin{bmatrix} x_{1i} \\ x_{2i} \\ x_{3i} \\ \vdots \\ x_{ni} \end{bmatrix} \text{)}$$

Consider a matrix P . The column vectors of P are n Eigen vectors such that

$$P = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{bmatrix} = \begin{bmatrix} \vec{x}_1 \\ \vec{x}_2 \\ \vdots \\ \vec{x}_n \end{bmatrix}$$

Let D is a diagonal matrix such that

(35)

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda_{nn} \end{bmatrix} = \text{diag}[\lambda_1, \lambda_2, \dots, \lambda_n]$$

The $PD :=$

$$\begin{bmatrix} \lambda_1 x_{11} & \lambda_2 x_{12} & \cdots & \lambda_n x_{1n} \\ \lambda_1 x_{21} & \lambda_2 x_{22} & \cdots & \lambda_n x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 x_{n1} & \lambda_2 x_{n2} & \cdots & \lambda_n x_{nn} \end{bmatrix}$$
$$= [\lambda_i x_{ij}]$$
$$= [\vec{A}\vec{x}_1, \vec{A}\vec{x}_2, \dots, \vec{A}\vec{x}_n]$$

$$PD = A[x_1, x_2, \dots, x_n] = AP$$

Now multiplying by P^{-1} ,

$$P^{-1}PD = P^{-1}AP$$

$$D = P^{-1}AP$$

Thus Pre multiplying A by P^{-1} and

Post multiplying by P we get the
 D diagonal matrix whose elements
in the diagonal are the Eigen values.

This process is said to be diagonalization
of matrix A .

Diagonalize the following matrices

(36)

$$(i) \begin{bmatrix} 4/3 & \sqrt{2}/3 \\ \sqrt{2}/3 & 5/3 \end{bmatrix}$$

$$(ii) \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(i) \text{ Let } A = \begin{bmatrix} 4/3 & \sqrt{2}/3 \\ \sqrt{2}/3 & 5/3 \end{bmatrix}$$

The characteristic equation is

$$(A - \lambda I) = \begin{vmatrix} 4/3 - \lambda & \sqrt{2}/3 \\ \sqrt{2}/3 & 5/3 - \lambda \end{vmatrix} = 0$$

$$(4/3 - \lambda)(5/3 - \lambda) - (\sqrt{2}/3)^2 = 0$$

$$\frac{20}{9} - \frac{4\lambda}{3} - \frac{5\lambda}{3} + \lambda^2 - \frac{2}{9} = 0$$

~~$$\cancel{\lambda^2} - \frac{9\lambda}{3} + \frac{18}{9} = 0$$~~

~~$$\lambda^2 - 3\lambda + 2 = 0$$~~

$$(\lambda - 1)(\lambda - 2) = 0$$

$$\lambda = 1, 2$$

The diagonalized matrix

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\left. \begin{array}{l} ax^2 + bx + c = 0 \\ x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ = \frac{3 \pm \sqrt{9 - 8}}{2} \\ x_1 = \frac{3 + 1}{2} = \frac{4}{2} = 2 \\ x_2 = \frac{3 - 1}{2} = \frac{2}{2} = 1 \\ (\lambda = 1, 2) \end{array} \right\}$$

~~(x₁ = 2, x₂ = 1)~~

$$(i) \text{ Let } A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(37)

The characteristic equation is

$$(A - \lambda I) = 0 = \begin{bmatrix} \cos\theta - \lambda & -\sin\theta & 0 \\ \sin\theta & \cos\theta - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{bmatrix}$$

$$(\cos\theta - \lambda) [(\cos\theta - \lambda)(1 - \lambda) - 0] - (-\sin\theta) [\sin\theta(1 - \lambda) - 0] = 0$$

$$\cancel{(\cos\theta - \lambda)(\cos\theta - \lambda)(1 - \lambda)} + \sin\theta \sin\theta(1 - \lambda) = 0$$

$$(\cos\theta - \lambda)^2 (1 - \lambda) + \sin^2\theta (1 - \lambda) = 0$$

$$(1 - \lambda) [(\cos\theta - \lambda)^2 + \sin^2\theta] = 0$$

$$(1 - \lambda) (\underline{\cos^2\theta + \lambda^2 - 2\cos\theta\lambda} + \underline{\sin^2\theta}) = 0$$

$$(1 - \lambda) (1 + \lambda^2 - 2\cos\theta\lambda) = 0$$

$$(1 - \lambda) (\lambda^2 - 2\cos\theta\lambda + 1) = 0$$

$$\cancel{(1 - \lambda)} \downarrow \quad \lambda_1 = 1 \quad \downarrow \quad \lambda_2 = e^{i\theta}, \quad \lambda_3 = e^{-i\theta}$$

$$D = \begin{bmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-i\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\lambda = \frac{-\cos\theta \pm \sqrt{\cos^2\theta - 4}}{2}$$

$$= \frac{\cos\theta \pm 2\sqrt{(\cos\theta - 1)}}{2}$$

$$= \frac{\cos\theta \pm 2\sqrt{(\cos^2\theta - 1)}}{2}$$

$$= \cos\theta \pm \sqrt{(\cos^2\theta - 1)}$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$(\cos^2\theta - 1) = -\sin^2\theta$$

$$= \cos\theta \pm \sqrt{(-1)\sin^2\theta}$$

$$= \cos\theta \pm i\sin\theta$$

$$= e^{\pm i\theta}$$

Similarity transformation

(38)

Consider $B = QAP \rightarrow (1)$

where P & Q are non-singular matrix

If $Q = P^{-1}$ then ~~it becomes~~ the form for
in (1) becomes ~~for~~ ~~transformation~~

$$\boxed{B = P^{-1}AP} \rightarrow (2)$$

This transformation of matrix A into B
is called similarity transformation and
the matrices A & B are called
similar matrices.

From (2) :

$$PBP^{-1} = P P^{-1}APP^{-1}$$

$$= I A I = A$$

$$\boxed{A = PBP^{-1}}$$

This equation also defines similarity
transformation.

A matrix equation retaining its form
under similarity transformation

Consider $AB = C \quad (3)$

$$P^{-1}(AB)P = P^{-1}CP$$

$$P^{-1}A(PP^{-1})BP = P^{-1}CP$$

↓

$$(P^{-1}AP)(P^{-1}BP) = P^{-1}CP$$

$$A' B' = C' \quad (4)$$

which is similar to eqn. (3)

If P of similarity transformation (39)

is unitary if it is called a unitary transformation.

If P is real orthogonal matrix it is called an orthogonal transformation.

Orthogonal and Unitary transformation.

Consider a linear transformation

$$Y = A X \quad (1)$$

$A \rightarrow$ Square matrix of order n

X & $Y \rightarrow$ Column vectors of order $n \times 1$

If A matrix is a unitary then the linear transformation (1) is called unitary transformation, where $A^T A = A A^T = I$.

From (1) $y^T y = (A x)^T A x$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
 $= x^T A^T A x$ $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$
 $y^T y = x^T x$ $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

(Or) $\sum_{i=1}^n y_i^* y_i = \sum_{i=1}^n x_i^* x_i$ (2)

If the linear transformation is further restricted to be real then it is called real orthogonal transformation.

From (2)

$$\begin{aligned} y^T y &= x^T x = x^T A^T A x \\ &= x^T x \quad (\text{real}) \quad (\cancel{(A^T A = I)}) \end{aligned}$$

$$Y = A \times \text{the condition for } A \text{ to be diagonal} \quad (4)$$

$$A^T A = A A^T = I$$

Special matrix.

① Hermitian & Skew matrix.

Show that every square matrix can be uniquely expressed as the sum of a Hermitian and skew-Hermitian matrix.

Let A be a square matrix.

A can be expressed as, $A = \frac{1}{2} A + \frac{1}{2} A^H + \frac{1}{2} A - \frac{1}{2} A^H$

$$A = \frac{1}{2} (A + A^H) + \frac{1}{2} (A - A^H) \quad (1)$$

where $A^H \rightarrow$ transpose conjugate of A .

$$\text{Let } P = \frac{1}{2} (A + A^H), Q = \frac{1}{2} (A - A^H)$$

① can be written as

$$A = P + Q \quad (2)$$

$$P^H = \left[\frac{1}{2} (A + A^H) \right]^H = \frac{1}{2} [A^H + (A^H)^H]$$

$$P^H = \frac{1}{2} (A^H + A) = P$$

$$\text{now } Q^H = \left[\frac{1}{2} (A - A^H) \right]^H = \frac{1}{2} [A^H - (A^H)^H]$$

$$Q^H = \frac{1}{2} (A^H - A)$$

$$= -\frac{1}{2} (A - A^H) = -Q$$

Thus P is Hermitian, Q is skew-Hermitian which shows that a square matrix

A can be represented as the sum of a Hermitian and skew-Hermitian matrix.

③ If H is Hermitian, Show that \underline{iH} is Skew-Hermitian. (1)

If H is Hermitian, then $H^T = H$ - ①

If iH is Skew-Hermitian, Then

$$(iH)^T = -iH \quad -\textcircled{2}$$

Now, $(iH)^T = i^* H^T = -iH^T$

$$(iH)^T = -iH \quad (H \text{ is Hermitian})$$

Thus $\textcircled{2}$ is satisfied $\xrightarrow{\text{condition for}} \text{Skew-Hermitian}$

④ If A is Hermitian, then $B^T A B$ is Hermitian for every matrix B .

If A is Hermitian, then $A^T = A$ - ①

If $B^T A B$ is Hermitian, then

$$(B^T A B)^T = B^T A B \rightarrow \textcircled{2}$$

Now, $(B^T A B)^T = B^T A^T (B^T)^T = B^T A^T B$
 $(B^T A B)^T = B^T \downarrow A^T B$

Thus eqn $\textcircled{2}$ is satisfied. Hence proved.

II Orthogonal matrix.

A square matrix A is said to be orthogonal if $A^T A = I$ - ①

$$A A^T = I \rightarrow \textcircled{2}$$

\downarrow Transpose of A unit matrix

We know $|A^T| = |A|^T = |A|$ and

$$\left| \frac{A^T}{A} \right| = \frac{|A^T|}{|A|} = 1$$

From ① if $A^T A = I$ which means (42)

~~$A^T A = I$~~

~~$|A^T A| = |A^T| / |A| = |I|$~~

IF $A^T A = I$

then $|A^T A| = |I| \neq 0$

$|A|^2 = I$

$A = \pm I$

\therefore The determinant of an orthogonal matrix can have only values ± 1 .

Also A is non-singular ($|A| \neq 0$)

$\therefore A^{-1}$ exists

$A^T A A^{-1} = I A^{-1}$

$A^T = A^{-1}$

Theorem

The product of orthogonal matrices are also orthogonal. If A and B are orthogonal then AB and BA are also orthogonal matrix.

Proof. If A and B are given orthogonal matrix,

Then $A^T A = I$

$B^T B = I$, such that $|A^T| = |A| \neq 0$

I f AB is orthogonal, then $(AB)^T = B^T A^T = B^T A = I$

$(AB)^T (AB) = (B^T A^T)(AB) = B^T (A^T A) B$

If BA is orthogonal $= B^T I B = B^T B = I$

Problem S.T. The following matrices are orthogonal (43)

$$(i) A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$(ii) A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(i) Solution

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$A^T A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

(ii) Solution

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A^T A =$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 1+0+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix} = I$$

Unitary matrices.

(44)

A square matrix is unitary if

$$A^T A = I$$

$$A A^T = I$$

$$(A^T)^*$$

we know, $(A^T)^* = |A|^{-1}$

$$(A^T A) = (A^T / |A|)$$

$$\Rightarrow (A^T A) = (I) = (A^T / |A|) = (I) = I$$

The determinant of a unitary matrix is always of unit modulus & hence a unitary matrix is non-singular

for 3.

The products of two unitary matrices are also unitary.

If A & B are unitary matrices, then $A^T B$ and $B^T A$ are also unitary.

$$A^T A = I, \quad B^T B = I, \quad \text{such that}$$

~~If~~ ^{the product} $A^T B$ is unitary, then

$$(AB)^+ (AB) = I$$

$$(B^T A^T)(AB) = I$$

$$B^T A^T A B = I$$

$$B^T I B = I$$

$$B^T B = I$$

Hence AB is unitary

$$A = \begin{bmatrix} -1 & i \\ 2 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} -i & 2 \\ 1 & 1 \end{bmatrix}$$

$$(A^T)^* = \begin{bmatrix} i & 2 \\ 1 & 1 \end{bmatrix}$$

$$|A|^2 = (-1)^2 + 2^2 = 5$$

$$A^* = \begin{bmatrix} i & 1 \\ 2 & 1 \end{bmatrix}$$

$$|A| = \sqrt{5}$$

Ques Show that the matrix $A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ (75) is unitary.

~~$$A^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$~~

$$(A^T)^* = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = A^+$$

$$A^+ A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + \frac{-i^2}{2} & \frac{i}{\sqrt{2}} \times \frac{i}{\sqrt{2}} + \frac{1}{2} \\ -\frac{i}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + \frac{i^2}{2} & -\frac{i}{\sqrt{2}} \times \frac{i}{\sqrt{2}} + \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} + \frac{1}{2} & 0 \\ 0 & \frac{1}{2} + \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

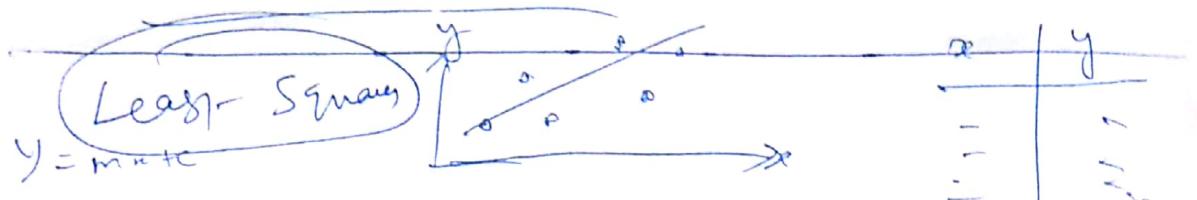
$$A^+ A = I$$

Hence the given matrix is unitary

Fourier and Laplace Transform

(46)

(eg) e.d & s. factor \rightarrow electron density &
periodic fns. Structure factor



$$y = f(x)$$

St. line should be fitted through the given pts.

(Sum of the square of the vertical deviation from the St. line should be least.)

principle of Least Squares.

$$S \rightarrow \text{Deviation.} \quad S = \sum_{i=1}^n v_i^2 = f(x_i) - y_i \quad v_i = c + mx_i - y_i$$

$$S = \sum_{i=1}^n [f(x_i) - y_i]^2$$

$$\frac{\partial S}{\partial m} = \sum_{i=1}^n [c + mx_i - y_i]^2$$

$$\frac{\partial S}{\partial c} = \sum_i (c + mx_i - y_i) = 0$$

$$\frac{\partial S}{\partial c} = \sum_i (c + mx_i - y_i) = 0$$

$$\sum_{i=1}^n x_i (c + mx_i - y_i) = 0$$

$$\left(\sum_{i=1}^n x_i \right) c + m \left(\sum_{i=1}^n x_i^2 \right) = \sum_{i=1}^n x_i y_i$$

$$\sum_{i=1}^n 1 \times 1 + m \sum_{i=1}^n x_i - \sum_{i=1}^n y_i = 0$$

$$n c + m \sum_{i=1}^n x_i = \sum_{i=1}^n y_i$$

Fourier transform.

Paris or f_{xyz} can be related by Fourier transform.

Example: Structure factor and Electron density
The generalized S-factor expression.

$$F_{hkl} = \sum_{i=1}^n f_i \exp(-2\pi i (hx + ky + lz))$$

f_i \downarrow
↳ atomic scattering factor

$h, k, l \rightarrow$ miller indices

$x, y, z \rightarrow$ atomic coordinates

$n \rightarrow$ total no. of atoms in the unit cell

$f_i \rightarrow$ atomic scattering factor

The electron density is given by,

$$\rho(xyz) = \frac{1}{V} \sum_i F_{hkl} \exp(2\pi i (hx + ky + lz))$$

\rightarrow the f_i is the scatterer, which scatters X-rays. The atom has electrons.

Depending on the no. of electrons, the diffraction intensity of X-rays varies.

The electron density of the atom decides the structure factor.

~~These two are~~
The F.T of structure factor gives the e.d. The I.F.T of e.d gives the structure factor.

In general a pair or function can be related as follows;

$$g(x) = \int_a^b f(t) k(x, t) dt$$

$g(x)$ is called "integral transform" or $f(x)$,
by the kernel $k(x, t)$.

- Several kernels are available in mathematics (Fig)
- $$\tilde{f}(\omega) = \int_0^L f(t) e^{-i\omega t} dt \rightarrow \text{Fourier transform}$$
- $$g(\omega) = \int_0^\infty f(t) e^{-\omega t} dt \rightarrow \text{Laplace transform}$$
- $$h(\omega) = \int_0^L f(t) t \ln(\omega t) dt \rightarrow \text{Hankel transform}$$
- $$m(\omega) = \int_0^L f(t) t^{\omega-1} dt \rightarrow \text{Mellin transform.}$$

Fourier transform.

If $f(x)$ is periodic fn of x , then
the Fourier integral of $f(x)$ is,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iwx} dw \int_{-\infty}^{+\infty} f(t) e^{-iwt} dt$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{iwx} dw \cancel{\int_{-\infty}^{+\infty} f(t) e^{-iwt} dt} \int_{-\infty}^{+\infty} f(t) e^{-iwt} dt$$

where

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{iwx} g(w) dw \quad \text{fn. of frequency}$$

where $g(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-iwt} dt$

$$\begin{cases} F = \omega \\ W = \omega \\ wt = \omega t \\ \frac{1}{\omega t} = \frac{1}{2\pi} \end{cases}$$

$g(w)$ is called the Fourier transform of $f(t)$ and $f(t)$ is called inverse transform or $\cancel{g(w)}$ for n. time

Fourier Sine and Cosine transforms.

The Fourier transform of $f(t)$ is $g(\omega)$
can be expressed as,

$$\begin{aligned}
 g(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt \\
 &= \frac{1}{\sqrt{2\pi}} \left[\underbrace{\int_{-\infty}^0 f(t) e^{-i\omega t} dt}_{\text{reflect by } t} + \int_0^{\infty} f(t) e^{-i\omega t} dt \right] \\
 &= \frac{1}{\sqrt{2\pi}} \left[\int_0^{\infty} f(-t) e^{i\omega t} dt + \int_0^{\infty} f(t) e^{-i\omega t} dt \right]
 \end{aligned}$$

$f(t) = f(-t)$ if $f(t)$ is even
 $f(t) = -f(-t)$ if $f(t)$ is odd.

$$\begin{aligned}
 g(\omega) &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(t) (e^{i\omega t} + e^{-i\omega t}) dt \\
 &\quad \hookrightarrow \text{for even } f(t) \\
 &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(t) (e^{-i\omega t} - e^{i\omega t}) dt \quad \text{for odd } f(t),
 \end{aligned}$$

$$\begin{aligned}
 \partial \frac{d}{dt} &= \cancel{f(t) e^{i\omega t}} + \cancel{-f(t) e^{-i\omega t}} \\
 &= \frac{1}{\sqrt{2\pi}} \left[\int_0^{\infty} -f(t) e^{i\omega t} dt + \int_0^{\infty} f(t) e^{-i\omega t} dt \right] \\
 &= \frac{1}{\sqrt{2\pi}} \left[f(t) [e^{-i\omega t} - e^{i\omega t}] \right]
 \end{aligned}$$

$$\frac{e^{i\omega t} + e^{-i\omega t}}{2} = \cos \omega t, \quad e^{i\omega t} - e^{-i\omega t} = 2i \sin \omega t$$

$$\frac{e^{i\omega t} - e^{-i\omega t}}{2i} = i \sin \omega t, \quad e^{i\omega t} + e^{-i\omega t} = 2 \cos \omega t$$

$$\therefore g(\omega) = \frac{1}{\sqrt{2\pi}} 2 \int_0^{\infty} f(t) \cos \omega t dt \quad \rightarrow \text{for even } f(t)$$

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \frac{2}{i} \int_0^{\infty} f(t) i \sin \omega t dt \quad \rightarrow \text{for odd } f(t)$$

These transforms are called

(50)

infinite cosine transform and infinite sine transform.

$$g_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos \omega t dt$$

$$g_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin \omega t dt$$

The inverse transforms are

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g_c(\omega) \cos \omega x d\omega$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g_s(\omega) \sin \omega x d\omega.$$

for periodic 1- Fourier transform.

1. Addition theorem or Linearity theorem.

If $f(t) = a_1 f_1(t) + a_2 f_2(t) + \dots$ then

the FT of $f(t)$ is given by,

$$g(\omega) = a_1 g_1(\omega) + a_2 g_2(\omega) + \dots$$

where $g_1(\omega), g_2(\omega) \dots$ are FTs of

$f_1(t), f_2(t) \dots$ and a_1, a_2, \dots are constants.

Proof.

The FT of $f(t)$ is given by

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} [a_1 f_1(t) + a_2 f_2(t) + \dots] e^{-i\omega t} dt$$

(51)

$$= a_1 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_1(t) e^{-i\omega t} dt +$$

$$a_2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_2(t) e^{-i\omega t} dt + \dots$$

$$= a_1 g_1(\omega) + a_2 g_2(\omega) + \dots$$

Q. Similarity theorem or change of ~~scale~~
property. If $\tilde{g}(\omega)$ is the FT of $f(t)$,
Then the FT of $f(at)$ is $\frac{1}{a} \tilde{g}\left(\frac{\omega}{a}\right)$.

PROOF

~~Fourier transform of~~ $[f(t)]$ is denoted by $\int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt = \tilde{g}(\omega)$

Let $y = at$ \Rightarrow $f(t) e^{-i\omega t} dt = f(y/a) e^{-i\omega y/a} dy$

$$FT [f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt = \tilde{g}(\omega)$$

$$FT [f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y/a) e^{-i\omega y/a} \frac{dy}{a}$$

$$FT [f(at)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(at) e^{-i\omega at} a dt$$

$$FT [f(at)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(a y/a) e^{-i\omega a y/a} a dy$$

$$FT f(at) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y) e^{-i\omega y} dy$$

$$FT[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt = g(\omega) \quad (5a)$$

$$FT[f(at)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(at) e^{-i\omega t} dt$$

Let $y = at$, $t = \frac{y}{a}$, $dt = \frac{dy}{a}$

$$\begin{aligned} FT[f(at)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y) e^{-i\omega y/a} \frac{dy}{a} \\ &= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y) e^{-i(\frac{\omega}{a})y} dy \\ &= \frac{1}{a} g\left(\frac{\omega}{a}\right) \end{aligned}$$

③ If $g(\omega)$ is the FT of $f(t)$,

then the FT of complex conjugate of $f(t)$ will be $g^*(-\omega)$, where $*$ \rightarrow complex conjugate.

Proof

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt$$

Take complex conjugate both sides

$$g^*(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f^*(t) e^{+i\omega t} dt$$

Replace ω by $-\omega$,

$$g^*(-\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f^*(t) e^{-i\omega t} dt$$

$$\therefore g^*(-\omega) = F.T [f^*(t)]$$

4. Shifting property

(53)

If $g(\omega)$ is the FT of $f(t)$, then the FT of $f(t \pm a)$ is $e^{\pm i\omega a} g(\omega)$, where a is a constant.

Proof.

$$FT \text{ of } f(t) =$$

$$FT [f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt$$

$$FT [f(t \pm a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t \pm a) e^{-i\omega t} dt$$

$$\text{Let } t \mapsto y = t \pm a, \quad t = y \mp a$$

$$dt = dy$$

$$FT [f(t \pm a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y) e^{-i\omega(y \mp a)} dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y) e^{-i\omega y} e^{\pm i\omega a} dy$$

$$= \cancel{e^{\pm i\omega a}} \underbrace{\int_{-\infty}^{+\infty} f(y) e^{-i\omega y} dy}_{\rightarrow g(\omega)}$$

$$FT [f(t \pm a)] = e^{\pm i\omega a} g(\omega)$$

5. Modulation theorem.

If $g(\omega)$ is the FT of $f(t)$, then the FT of $f(t) \cos \omega_0 t$ is $\frac{1}{2} g(\omega - \omega_0) + \frac{1}{2} g(\omega + \omega_0)$

Proof.

$$FT [f(t) \cos \omega_0 t] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) \cos \omega_0 t e^{-i\omega t} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) \left(\frac{e^{iat} + e^{-iat}}{2} \right) e^{i\omega t} dt \quad (5f)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i^*(\omega - a)t} f(t) dt + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i(\omega+a)t} f(t) dt$$

$$= \frac{1}{2} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i^*(\omega - a)t} f(t) dt + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i(\omega+a)t} f(t) dt \right]$$

$$= \frac{1}{2} [g(\omega - a) + g(\omega + a)]$$

(6) Convolution theorem.

The transform of a product of two functions is given by the convolution integral

Proof Let $f_1(t)$ and $f_2(t)$ be two given fn.

Their product is $f(t) = f_1(t) f_2(t)$

$$FT[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_1(t) f_2(t) e^{-i\omega t} dt$$

Let $\underline{g}_1(\omega) = FT[f_1(t)] = \int_{-\infty}^{+\infty} f_1(t) e^{-i\omega t} dt$
 $\underline{g}_2(\omega) = FT[f_2(t)] = \int_{-\infty}^{+\infty} f_2(t) e^{-i\omega t} dt$
 \therefore Fourier inverse transform of $\underline{g}_1(\omega)$ will give $f_1(t)$

$$f_1(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \underline{g}_1(\omega') e^{i\omega' t} d\omega'$$

note $FT[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \underline{g}_1(\omega') e^{i\omega' t} d\omega' \right] f_2(t) e^{-i\omega t} dt$

$$= \frac{1}{2\pi i} \int_{-\infty}^{+\infty} g_1(\omega') \frac{1}{2\pi i} \int_{-\infty}^{+\infty} f_2(t) e^{-i(\omega - \omega') t} dt d\omega$$

we have

$$g_2(\omega) = \int_{-\infty}^{+\infty} f_2(t) e^{-i\omega t} dt$$

Replace ω by $\omega - \omega'$,

$$f_2(\omega - \omega') = \int_{-\infty}^{+\infty} f_2(t) e^{-i(\omega - \omega') t} dt$$

$$\therefore FT[f(t)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g_1(\omega') g_2(\omega - \omega') d\omega'$$

The FT of a product of two functions $f_1(t)$ and $f_2(t)$ is given by the integral known as convolution integral, in which g_1 and g_2 convolve with each other.

(7) Parseval's theorem:

The FT of a convolution integral is given by the product of transforms of the convolving functions.

Proof: Let $f(t)$ is the convolution integral.

$$f(t) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} f_1(t') f_2(t-t') dt'$$

The FT of $f(t)$ is $\tilde{g}(\omega)$

$$\tilde{g}(\omega) = FT[f(t)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} f_1(t') f_2(t-t') dt' e^{-i\omega t} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_1(t') e^{-i\omega t'} dt' \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_2(t-t') e^{-i\omega t} e^{i\omega t'} dt' \xrightarrow{\text{multiply by } \frac{e^{-i\omega t}}{e^{i\omega t}}} e^{-i\omega(t-t')}$$

If $f_1(\omega)$ and $f_2(\omega)$ are the
FT of $f_1(t)$ and $f_2(t)$ respectively.

then $f_1(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_1(t') e^{-i\omega t'} dt'$

$$f_2(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_2(t) e^{-i\omega t} dt$$

~~$f_1(\omega) = f_2(\omega)$~~ replace t by $t-t'$

$$f_2(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_2(t-t') e^{-i\omega(t-t')} dt$$

$$\therefore g(\omega) = f_1(\omega) \cdot f_2(\omega)$$

(8) The Fourier transform of the
square modulus of a fn. is given by
self convolution integral.

Proof: let $f(t) = |\phi(t)|^2 = \phi(t) \phi^*(t)$

where $\phi(t)$ is the complex conjugate of $\phi^*(t)$.

The FT of $f(t)$ is $\text{FT}[f(t)]$

$$\text{FT}[f(t)] = g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(t) \phi^*(t) e^{-i\omega t} dt$$

$$\text{The FT of } \phi(t) = \psi(\omega') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(t) e^{i\omega' t} dt$$

The inverse FT or $\psi(\omega')$ is $\phi(t)$ (57)

$$\therefore \phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \psi(\omega') e^{i\omega' t} d\omega'$$

$$\begin{aligned} \therefore g(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} [\psi(\omega') e^{i\omega' t} d\omega'] \phi^*(t) e^{-i\omega t} dt \right] d\omega' \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \underbrace{\psi(\omega')}_{\omega} \underbrace{\phi^*(t)}_{\omega'} e^{-i(\omega - \omega')t} dt d\omega' \end{aligned}$$

The complex conjugate of $\psi(\omega')$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi^*(t) e^{i\omega' t} dt = \psi^*(\omega')$$

Replace ω' by $\omega' - \omega$,

$$\psi^*(\omega' - \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi^*(t) e^{i(\omega' - \omega)t} dt$$

$$\therefore g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \psi(\omega') \psi^*(\omega' - \omega) d\omega'$$

(9) Derivative of F.T.

If $g(\omega)$ is the FT of $f(t)$, then

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt$$

Differentiate both sides w.r.t ω .

$$\begin{aligned} \frac{dg(\omega)}{d\omega} &= \frac{1}{\sqrt{2\pi}} \frac{d}{d\omega} \left[\int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt \right] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\partial}{\partial \omega} [f(t) e^{-i\omega t}] dt \end{aligned}$$

(58)

$$= \frac{1}{\sqrt{2\pi}} (-it) \int_{-\infty}^{+\infty} f(t) e^{-iwt} dt$$

$$= \frac{-i}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} t f(t) e^{-iwt} dt$$

$$\frac{dg}{dw} = -i \text{FT} [t f(t)]$$

Differentiate n times w.r.t w , we get

$$\frac{d^n g(w)}{dw^n} = (-i)^n \text{FT} [t^n f(t)]$$

$$\therefore \text{Re } \text{FT} \left[i \frac{dg}{dw} \right] = \text{FT} [t f(t)]$$

\therefore The F.T of $t f(t)$ is $i \frac{dg}{dw}$

The inverse transform of $\frac{dg}{dw}$ is $-i t f(t)$

The F.T of $\left[(-it)^n f(t) \right]$ is $\frac{d^n g}{dw^n}$

The inverse transform of $\frac{d^n g}{dw^n}$ is $(-it)^n f(t)$

(10) Fourier transform of a derivative

Let $g(w) = \text{FT} [f(t)]$, $g(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-iwt} dt$

Let $g_1(w) = \text{FT} \left[\frac{df(t)}{dt} \right]$, $g_1(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{df(t)}{dt} e^{-iwt} dt$

Integrating by parts

$$g_1(w) = \frac{1}{\sqrt{2\pi}} \left[\left[e^{-iwt} f(t) \right]_{-\infty}^{+\infty} - (-iw) \int_{-\infty}^{+\infty} f(t) e^{-iwt} dt \right]$$

$$g_1(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\omega t} f(t) dt + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt \quad (59)$$

$$g_1(\omega) = \frac{1}{\sqrt{2\pi}} \left[e^{-i\omega t} f(t) \right]_{-\infty}^{+\infty} + i\omega g(\omega)$$

As $t \rightarrow \infty$, $e^{-i\omega t} \rightarrow 0$ ($e^{-i\omega t} = \frac{1}{e^{i\omega t}}$)

At $t \rightarrow -\infty$, $e^{-i\omega t} \rightarrow \infty$

\therefore The fn $f(t)$ should be well behaved in:

If f should decrease to 0 when $t \rightarrow \infty$

we have, $\dot{g}_1(\omega) = i\omega g(\omega)$

$$\therefore FT \left[\frac{df}{dt} \right] = i\omega FT [f(t)]$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{df}{dt} e^{-i\omega t} dt = i\omega \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt$$

Replace, $f(t)$ by $\frac{d^2 f}{dt^2}$, both sides

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{d^2 f}{dt^2} e^{-i\omega t} dt = i\omega \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{df}{dt} e^{-i\omega t} dt \\ = (i\omega)^2 g(\omega)$$

$$\therefore FT \left[\frac{d^2 f}{dt^2} \right] = (i\omega)^2 FT [f(t)].$$

Repeating the process n times,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{d^n f}{dt^n} e^{-i\omega t} dt = (i\omega)^n g(\omega)$$

$$(or) \quad FT \left[\frac{d^n f}{dt^n} \right] = (i\omega)^n FT [f(t)]$$

(11) Fourier sine and cosine transforms

(60)

or derivatives

The F. sine transform is,

$$g_s(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin wt dt$$

$$g_c(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos wt dt$$

now consider a well behaved function such

$f(t)$ & its derivatives $\rightarrow 0$, as $t \rightarrow \infty$.

The F. sine transform or first derivative

$$\text{i.e., } g_{1s}(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{df}{dt} \sin wt dt$$

$$= \sqrt{\frac{2}{\pi}} \left[f(t) \sin wt \right]_0^{\infty} - \sqrt{\frac{2}{\pi}} w \int_0^{\infty} f(t) \cos wt dt$$

when $t \rightarrow \infty$, $f(t) \rightarrow 0$

$\therefore f(t)$ in first term = 0

$$\therefore g_{1s}(w) = - \sqrt{\frac{2}{\pi}} w \int_0^{\infty} f(t) \cos wt dt$$

$$g_{1s}(w) = - w g_c(w)$$

Fourier cos transform

$$g_{1c}(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{df}{dt} \cos wt dt$$

$$= \sqrt{\frac{2}{\pi}} \left[f(t) \cos wt \right]_0^{\infty} + \sqrt{\frac{2}{\pi}} w \int_0^{\infty} f(t) \sin wt dt$$

$$= - \sqrt{\frac{2}{\pi}} f(0) + \cancel{w} g_s(w)$$

$$= - \sqrt{\frac{2}{\pi}} f(0) + w g_s(w)$$

$$= w g_s(w) - \sqrt{\frac{2}{\pi}} f(0)$$

$$\begin{aligned} \int u dv &= uv - \int v du \\ \int u dv &= uv - \int v du \end{aligned}$$

The FT of second derivative $\frac{d^2 f}{dt^2}$ is

(61)

$$g_{2s}(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{df}{dt^2} \sin wt dt$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{df}{dt} \sin wt \right]_0^\infty - \sqrt{\frac{2}{\pi}} w \int_0^\infty \frac{df}{dt} \cos wt dt$$

$\rightarrow 0$ when $t \rightarrow \infty$

$$g_{2s}(w) = -w g_{1c}(w)$$

$$= -w [w g_s(w) - \sqrt{\frac{2}{\pi}} f(0)]$$

$$g_{2s}(w) = -w^2 g_s(w) + \sqrt{\frac{2}{\pi}} w f(0)$$

The F_1^{Cofine} or $\underline{\text{II}}$ derivative is

$$g_{dc}(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{df'}{dt^2} \cos wt dt$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{df}{dt} \cos wt \right]_0^\infty + \sqrt{\frac{2}{\pi}} w \int_0^\infty \frac{df}{dt} \sin wt dt$$

$$= -\sqrt{\frac{2}{\pi}} \frac{df(0)}{dt} + w g_{1s}(w)$$

$$= -\sqrt{\frac{2}{\pi}} f'(0) - w g_{1s}(w)$$

$$\boxed{g_{dc}(w) = -\sqrt{\frac{2}{\pi}} f'(0) - w^2 g_s(w)}$$

Problem:

1)

Find the FT of the slit fn $f(x)$, given by

$$f(x) = \begin{cases} 1/e & |x| \leq \epsilon \\ 0 & |x| > \epsilon \end{cases}$$

Determine the limit of this transform as $\epsilon \rightarrow 0$.

Sohwan

(62)

$$\begin{aligned}
 & \text{The FT of a fn. } f(x) \text{ is } g(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-iwx} dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\epsilon}^{+\epsilon} \frac{1}{\epsilon} e^{-iwx} dx = \frac{1}{\sqrt{2\pi}\epsilon} \left[\frac{e^{-iwx}}{-iw} \right]_{-\epsilon}^{+\epsilon} \\
 &= \frac{1}{\sqrt{2\pi}} \cancel{\frac{1}{\epsilon} \left[e^{iwx} - e^{-iwx} \right]}_{iw} \\
 &= \frac{1}{\sqrt{2\pi}} \frac{1}{\epsilon} \left[\frac{e^{-iwe} - e^{iwe}}{-iw} \right] \\
 &= \frac{1}{\sqrt{2\pi}} \frac{1}{\epsilon} \left[\frac{e^{iwe} - e^{-iwe}}{iw} \right] \\
 &= \frac{1}{\sqrt{2\pi}} \frac{1}{w\epsilon} \left[\frac{e^{iwe} - e^{-iwe}}{i} \right] \\
 &= \frac{2}{\sqrt{2\pi}} \frac{1}{w\epsilon} \left[\frac{e^{iwe} - e^{-iwe}}{2i} \right] \\
 &= \sqrt{\frac{2}{\pi}} \frac{1}{w\epsilon} \sin w\epsilon = \sqrt{\frac{2}{\pi}} \left[\frac{\sin w\epsilon}{w\epsilon} \right]
 \end{aligned}$$

$$\begin{aligned}
 & \lim_{\epsilon \rightarrow 0} g(w) = \lim_{\epsilon \rightarrow 0} \sqrt{\frac{2}{\pi}} \frac{\sin w\epsilon}{w\epsilon} \\
 &= \lim_{\epsilon \rightarrow 0} \sqrt{\frac{2}{\pi}} \frac{\frac{\partial}{\partial \epsilon} (\sin w\epsilon)}{\frac{\partial}{\partial \epsilon} (w\epsilon)} = \lim_{\epsilon \rightarrow 0} \sqrt{\frac{2}{\pi}} \frac{w \cos w\epsilon}{w} \\
 &= \sqrt{\frac{2}{\pi}}
 \end{aligned}$$

The FT of $g(w)$ approaches $\sqrt{\frac{2}{\pi}}$ as $\epsilon \rightarrow 0$.

Find the FT of the fn $f(x) = N e^{-\alpha x^2}$ (63)

$$\begin{aligned}
 g(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\omega x} dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} N e^{-\alpha x^2} e^{-i\omega x} dx \\
 &= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\left(\alpha x^2 + i\omega x\right)} dx \\
 &= \frac{N}{\sqrt{2\pi}} e^{-\alpha \left[x^2 + \frac{i\omega x}{\alpha} + \left(\frac{\omega}{2\alpha}\right)^2\right]} \cdot e^{\alpha \left(\frac{\omega}{2\alpha}\right)^2} dx \\
 &= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\left(\alpha x^2 + i\omega x\right)} e^{\alpha \left(\frac{\omega}{2\alpha}\right)^2} e^{-\alpha \left(\frac{\omega}{2\alpha}\right)^2} dx \\
 &= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\alpha \left[x^2 + \frac{i\omega x}{\alpha} + \left(\frac{\omega}{2\alpha}\right)^2\right]} e^{\alpha \left(\frac{\omega}{2\alpha}\right)^2} dx
 \end{aligned}$$

$$\begin{aligned}
 &\text{Diagram: A complex plane with a contour } C \text{ consisting of a large circle of radius } R \text{ and a small circle of radius } r. \text{ The contour is oriented counter-clockwise.} \\
 &= \frac{N}{\sqrt{2\pi}} e^{-\omega^2/4\alpha} \int_{-\infty}^{+\infty} e^{-\alpha \left(x^2 + \frac{i\omega x}{\alpha} - \left(\frac{\omega}{2\alpha}\right)^2\right)} dx \\
 &= \frac{N}{\sqrt{2\pi}} e^{-\omega^2/4\alpha} \int_{-\infty}^{+\infty} e^{-\alpha \left(x + \frac{i\omega}{2\alpha}\right)^2} dx
 \end{aligned}$$

$$\begin{aligned}
 \left(\frac{\omega}{2\alpha}\right)^2 &= \frac{\omega^2}{4\alpha^2} \\
 -\alpha \left(\frac{\omega}{2\alpha}\right)^2 &= -\alpha \frac{\omega^2}{4\alpha^2} \\
 e^{-\alpha \left(\frac{\omega}{2\alpha}\right)^2} &= e^{-\frac{\omega^2}{4\alpha^2}} \\
 \boxed{=} & \\
 \boxed{=} &
 \end{aligned}$$

$$let -y = x + \frac{i\omega}{2\alpha}$$

$$dy = dx, \quad y^2 = \left(x + \frac{i\omega}{2\alpha}\right)^2$$

~~$$e^{-\alpha \left(x^2 + \frac{i\omega x}{\alpha} - \left(\frac{\omega}{2\alpha}\right)^2\right)}$$~~

~~$$= e^{-\alpha \left(x^2 + \frac{i\omega x}{\alpha} - \left(\frac{\omega}{2\alpha}\right)^2\right)}$$~~

$$x - \frac{i\omega}{2\alpha}, \quad x + \frac{i\omega}{2\alpha}$$

$$-\frac{i\omega x}{\alpha} + \left(\frac{\omega}{2\alpha}\right)^2 + \frac{i\omega x}{\alpha} + \alpha x$$

$$= e^{-\frac{\omega^2}{4\alpha}}$$

$$\begin{aligned}
 &\cancel{x + \frac{i\omega}{2\alpha}} - \cancel{i\omega x} \\
 &= x^2 + \frac{\omega^2}{4\alpha^2} \\
 &= x^2 + \left(\frac{\omega}{2\alpha}\right)^2 \\
 &= x^2 + \frac{\omega^2}{4\alpha^2}
 \end{aligned}$$

(64)

$$\begin{aligned} & x + \frac{i\omega}{2\alpha} \\ & x + i\frac{\omega}{2\alpha} \\ \hline & x^2 + \frac{i\omega x}{\alpha} + \frac{i\omega x}{\alpha} - \left(\frac{\omega}{2\alpha}\right)^2 \end{aligned}$$

$$\hline x^2 + \frac{2i\omega x}{\alpha} - \left(\frac{\omega}{2\alpha}\right)^2$$

$$\begin{aligned} & = \frac{N}{\sqrt{\alpha\pi}} e^{-\omega^2/4\alpha} \int_{-\infty}^{+\infty} e^{-\alpha y^2} dy \\ & = \int_{-\infty}^{\infty} \frac{N}{\sqrt{\alpha\pi}} e^{-\omega^2/4\alpha} \sqrt{\frac{\pi}{\alpha}} = \cancel{\frac{\pi}{\alpha}} \\ & = \frac{\sqrt{\pi}}{\sqrt{\alpha\pi}} \frac{\sqrt{\pi}}{\sqrt{\alpha}} e^{-\omega^2/4\alpha} = \boxed{\sqrt{\frac{\pi}{2\alpha}} e^{-\omega^2/4\alpha}} \end{aligned}$$

Find the FT of $e^{-|t|}$

Solution

$$\begin{aligned} & \text{FT or } e^{-|t|} = g(\omega) \\ & g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-|t|} e^{-i\omega t} dt \\ & = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^0 e^{-|t|} e^{-i\omega t} dt + \int_0^{\infty} e^{-|t|} e^{-i\omega t} dt \right] \\ & = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^0 e^t e^{-i\omega t} dt + \int_0^{\infty} e^{-t} e^{-i\omega t} dt \right] \\ & \quad \begin{array}{l} \text{from } -\infty \text{ to } 0 \\ e^{-|t|} \rightarrow e^{-t} \end{array} \quad \begin{array}{l} \text{can't exist} \\ \text{only } e^t \text{ exists.} \end{array} \\ & = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^0 e^{t(1-i\omega)} dt + \int_0^{\infty} e^{-t(1+i\omega)} dt \right] \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{t(1-i\omega)}}{1-i\omega} \right]_0^\infty + \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-t(1+i\omega)}}{1+i\omega} \right]_0^\infty \quad (65) \\
 &= \frac{1}{\sqrt{2\pi}} \left[\frac{e^0(1-i\omega)}{(1-i\omega)} - \frac{e^{-\infty(1-i\omega)}}{(1-i\omega)} \right] + \left[\frac{e^{-\infty(1+i\omega)}}{(1+i\omega)} - \frac{e^0(1+i\omega)}{(1+i\omega)} \right] \\
 &= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{(1-i\omega)} \right] + \frac{1}{\sqrt{2\pi}} \left[\frac{1}{1+i\omega} \right] \\
 &= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{(1+i\omega)} + \frac{1}{(1-i\omega)} \right] = \frac{1}{\sqrt{2\pi}} \frac{(1+i\omega) + (1-i\omega)}{(1+i\omega)(1-i\omega)} \\
 &= \frac{1}{\sqrt{2\pi}} \left[\frac{2}{1+\omega^2} \right] = \frac{\sqrt{2}\sqrt{\pi}}{\sqrt{2}\sqrt{\pi}} \left(\frac{1}{1+\omega^2} \right) \\
 &= \sqrt{\frac{2}{\pi}} \left(\frac{1}{1+\omega^2} \right)
 \end{aligned}$$

Write the FT or the for $f(t)$ and prove moment theorem (i.e.)

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{m_n}{n!} (-i\omega)^n$$

where $m_n = \int_{-\infty}^{+\infty} t^n f(t) dt$ known as moment of $f(t)$.

Solution:

$$\begin{aligned}
 g(\omega) &= \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt = \int_{-\infty}^{+\infty} f(t) \left[1 - i\omega t - \frac{(i\omega t)^2}{2!} - \frac{(i\omega t)^3}{3!} + \dots \right] dt \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) \sum_{n=0}^{\infty} \frac{(-i\omega t)^n}{n!} dt \\
 &= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-i\omega)^n}{n!} \int_{-\infty}^{+\infty} t^n f(t) dt \\
 &= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{m_n}{n!} (-i\omega)^n
 \end{aligned}$$

Find the sine transform of $f(x)$. $f(x) = e^{-ax}$ (66)

$$g_s(w) = \sqrt{\frac{a}{\pi}} \int_0^\infty f(x) \sin wx dx$$

$$= \sqrt{\frac{a}{\pi}} \int_0^L \frac{e^{-ax}}{x} \sin wx dx$$

Differentiate w.r.t. w ,

$$\frac{dg_s(w)}{dw} = \sqrt{\frac{a}{\pi}} \int_0^L \frac{e^{-ax}}{x} x \cos wx dx$$

$$= \sqrt{\frac{a}{\pi}} \int_0^L e^{-ax} \cos wx dx = \sqrt{\frac{a}{\pi}} \left(\frac{a}{ax+w^2} \right)$$

Integrating,

$$g_s(w) = \sqrt{\frac{a}{\pi}} \tan^{-1}\left(\frac{w}{a}\right) + A, \text{ constant of integration.}$$

$$\text{For } w=0, \quad g_s(w) = g_s(0) = A.$$

But $\underline{g_s(w) = 0}$ for $w=0$
 $\therefore A=0$

$$\therefore g_s(w) = \sqrt{\frac{a}{\pi}} \tan^{-1}\left(\frac{w}{a}\right)$$

① Find the cosine transform of $f(x)$, which is unity for $0 < x < a$ and 0 for $x \geq a$.

② What is the $f(x)$ whose cosine transform is

$$\sqrt{\frac{a}{\pi}} \frac{\sin ap}{p}.$$

③ Solution $f(x) = \begin{cases} 1 & \text{for } 0 < x < a \\ 0 & x \geq a \end{cases}$

$$g_s(w) = \sqrt{\frac{a}{\pi}} \int_0^L f(x) \cos wx dx$$

$$= \sqrt{\frac{a}{\pi}} \int_0^a f(x) \cos wx dx + \int_a^L f(x) \cos wx dx$$

(67)

$$= \int_{\frac{\pi}{\omega}}^{\infty} \left[\int_0^x 1 \cdot \cos \omega x \, dx + \int_a^x 0 \cdot \cos \omega x \, dx \right] \, d\omega$$

$$= \int_{\frac{\pi}{\omega}}^{\infty} \frac{\sin \omega x}{\omega} \Big|_0^x \, d\omega = \int_{\frac{\pi}{\omega}}^{\infty} \frac{\sin \omega x}{\omega} \, d\omega$$

Q Given $g(\theta) = \int_{\frac{\pi}{\omega}}^{\infty} \frac{\sin \omega x}{\omega} \, d\omega$

or $g(\omega) = \int_{\frac{\pi}{\omega}}^{\infty} \frac{\sin \omega x}{\omega} \, dx$

The F. inverse cosine transform is

$$f(x) = \int_{\frac{\pi}{\omega}}^{\infty} g(\omega) \cos \omega x \, d\omega$$

$$= \int_{\frac{\pi}{\omega}}^{\infty} \int_{\frac{\pi}{\omega}}^{\infty} \frac{\sin \omega x}{\omega} (\sin \omega a) \cos \omega x \, d\omega$$

$$= \frac{1}{\pi} \int_{\frac{\pi}{\omega}}^{\infty} \frac{\sin((a+x)\omega) + \sin(a-x)\omega}{\omega} \, d\omega$$

$$= \frac{1}{\pi} \left[\int_0^{\infty} \left[\frac{\sin((a+x)\omega)}{\omega} + \frac{\sin(a-x)\omega}{\omega} \right] \, d\omega \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi}{2} + \frac{\pi}{2} \right] \text{ if } x < a$$

$$= \frac{1}{\pi} \left[\frac{\pi}{2} - \frac{\pi}{2} \right] \text{ if } x > a$$

$$f(x) = \begin{cases} 1 & \text{if } x < a \\ 0 & \text{if } x > a \end{cases}$$

$\sin(A+B)$
$\sin(A-B)$
$\sin \cos A + \cos \sin A$
$\sin \cos A - \cos \sin A$

$$\sin(\omega x + \omega t) = \sin \omega t \cos \omega x + \cos \omega t \sin \omega x$$

$$\sin(\omega x - \omega t) = \sin \omega t \cos \omega x - \cos \omega t \sin \omega x$$

$$\sin(\omega x + \omega t) + \sin(\omega x - \omega t) = 2 \sin \omega t \cos \omega x$$

$$\sin(\omega x + \omega t) + \sin(\omega x - \omega t) = 2 \sin \omega t \cos \omega x$$

q

The Laplace transform.

(68)

The LT of a fn. $F(t)$ is $\underline{f(s)}$
 (denoted by $L\{F(t)\}$) & defined as

$$f(s) = L\{f(t)\} = \lim_{a \rightarrow \infty} \int_a^{\infty} e^{-st} F(t) dt$$

$$= \int_0^{\infty} e^{-st} F(t) dt$$

L is called Laplace transform operator.

s may be real or complex no.

(but generally it is taken as real no.).

The LT of a fn. $F(t)$ exists only if the fn. satisfies the following conditions.

(1) The function $F(t)$ should be an arbitrary piecewise continuous fn. in every finite interval and $F(t) = 0$ for all negative values of t .

(2) The fn. $F(t)$ should be of exponential order.

This means

$$\left\{ \begin{array}{l} \text{if } \int_0^{\infty} e^{-st} F(t) e^{st} dt \text{ is finite} \\ \text{for all } s \end{array} \right. \\ = \int_0^{\infty} e^{-st} M dt$$

for some value of $s = s_0$
and M is a positive

constant (finite no.)

i.e. if $|F(t)| e^{st} \leq M$, then $F(t)$ is of exponential order so

said to be of exponential order so

properties of Laplace transform.

1. Linearity property

If a_1 and a_2 are constants and (69)
 the LT of $F_1(t)$ and $F_2(t)$ are
 $f_1(s)$ and $f_2(s)$ respectively. Then the
 Laplace transform of $a_1 F_1(t) + a_2 F_2(t)$,
 is $a_1 f_1(s) + a_2 f_2(s)$.

(i.) $\mathcal{L}\{a_1 F_1(t) + a_2 F_2(t)\} = a_1 \mathcal{L}\{F_1(t)\}$
 +
 $a_2 \mathcal{L}\{F_2(t)\}$

Proof. $\mathcal{L}\{F_1(t)\} = f_1(s) = \int_0^{\infty} e^{-st} F_1(t) dt$

$\mathcal{L}\{F_2(t)\} = f_2(s) = \int_0^{\infty} e^{-st} F_2(t) dt$

$\therefore \mathcal{L}\{a_1 F_1(t) + a_2 F_2(t)\} = \int_0^{\infty} e^{-st} [a_1 F_1(t) + a_2 F_2(t)] dt$
 = $a_1 \int_0^{\infty} e^{-st} F_1(t) dt + a_2 \int_0^{\infty} e^{-st} F_2(t) dt$
 = $a_1 f_1(s) + a_2 f_2(s)$

Generalizing

$$\mathcal{L}\left\{\sum_{m=1}^n a_m F_m(t)\right\} = \sum_{m=1}^n a_m \mathcal{L}\{F_m(t)\}$$

2. Change of Scale property

If $f(s)$ is the LT of $F(t)$, then the
 LT of $F(at)$ is $\frac{1}{a} f(\frac{s}{a})$.

Proof $\mathcal{L}\{F(t)\} = f(s) = \int_0^{\infty} e^{-st} F(t) dt$

$$\mathcal{L} \{ F(at) \} = \int_0^{\infty} F(at) e^{-st} dt$$

$$dt = at = u, \frac{dt}{a} = du, dt = \frac{du}{a}$$

$$\mathcal{L} \{ F(at) \} = \int_0^{\infty} F(u) e^{-su/a} \frac{du}{a}$$

$$= \frac{1}{a} \int_0^{\infty} F(u) e^{-\left(\frac{s}{a}\right)u} du = \frac{1}{a} f\left(\frac{s}{a}\right)$$

3. First translation (Shifting property)

If $f(s)$ is the LT of $F(t)$, then

LT of $e^{at} F(t)$ will be $f(s-a)$.

Proof:

$$\mathcal{L} \{ F(t) \} = f(s) = \int_0^{\infty} e^{-st} F(t) dt$$

$$\mathcal{L} \{ e^{at} F(t) \} = \int_0^{\infty} e^{-st} e^{at} F(t) dt$$

$$= \int_0^{\infty} F(t) e^{-(s-a)t} dt = f(s-a)$$

Why we can show, $f(s+a) = \mathcal{L} \{ e^{at} F(t) \}$

4. Second translation property

If $\mathcal{L} \{ F(t) \} = f(s)$, and $g(t) = \begin{cases} 0 & \text{if } t < a \\ F(t-a) & \text{if } t > a \end{cases}$

Then $\mathcal{L} \{ g(t) \} = e^{-as} f(s)$.

Proof:

$$\mathcal{L} \{ g(t) \} = \int_0^{\infty} e^{-st} g(t) dt$$

$$= \int_0^{\infty} e^{-st} h(t) dt + \int_a^{\infty} e^{-st} g(t) dt. \quad (7)$$

$$= \int_0^a e^{-st} \circ dt + \int_a^{\infty} e^{-st} F(t=a) dt$$

$$= 0 + \int_a^{\infty} e^{-st} F(t=a) dt$$

Let $t-a = \beta$ $dt = d\beta$

$$\begin{aligned} L\{g(t)\} &= \int_a^{\infty} e^{-s(\beta+a)} F(\beta) d\beta \\ &= e^{-sa} \int_a^{\infty} e^{-s\beta} F(\beta) d\beta = e^{-sa} f(s) \end{aligned}$$

5. Derivative of Laplace transform.

If $f(s)$ is the $L\{f\} = F(t)$, then

$$f'(s) = \frac{df}{ds} = L\{-t F(t)\}$$

and in general $L\{t^n F(t)\}$

$$= (-1)^n f^{(n)}(s) = (-1)^n \frac{d^n f(s)}{ds^n}$$

Proof

$$f(s) = \int_0^{\infty} e^{-st} F(t) dt$$

Differentiate both sides w.r.t. s ,

$$\begin{aligned} f'(s) &= \frac{df}{ds} = \int_0^{\infty} -t e^{-st} F(t) dt \\ &= \int_0^{\infty} e^{-st} [-t F(t)] dt \end{aligned}$$

$$\therefore f'(s) = L\{-t F(t)\}$$

Differentiating n times

(7)

$$f^n(s) = \frac{d^n f(s)}{ds^n} = L \left\{ (-1)^n t^n F(t) \right\}$$
$$= (-1)^n L \left\{ t^n F(t) \right\}$$

Problems

Find the LT of -

(i) k, a constant. (ii) t (iii) kt

(iv) t^n , $n \geq 0$ (v) e^{at} (vi) e^{-at}

(i) $F(t) = k$

$$f(s) = L \{k\} = \int_0^{\infty} e^{-st} k dt = k \left[\frac{e^{-st}}{-s} \right]_0^{\infty}$$
$$= \frac{k e^{-s \cdot 0}}{-s} - \frac{k e^{\infty}}{-s}$$

(ii)

$$F(t) = t$$
$$L(t) = \int_0^{\infty} e^{-st} t dt = t \left[\frac{e^{-st}}{-s} \right]_0^{\infty} - \int_0^{\infty} \frac{e^{-st}}{-s} dt$$
$$\left(\equiv u v - \int u dv \right)$$
$$= \frac{t^0}{s} - \left[\frac{1}{-s} \left[\frac{e^{-st}}{(-s)} \right] \right]_0^{\infty}$$
$$= \frac{1}{s^2}$$

(iii)

$$F(t) = kt$$
$$L \{kt\} = \int_0^{\infty} e^{-st} kt dt = k \int_0^{\infty} e^{-st} t dt$$
$$= k \frac{1}{s^2} = \frac{k}{s^2}$$

(iv) $F(t) = t^n$, $n \geq 0$

$$L \{t^n\} = \int_0^{\infty} t^n e^{-st} dt$$

We know, $\int_0^L t e^{-st} dt = \frac{1}{s^2} \left(\frac{1!}{s^{1+1}} \right) \quad (73)$

Differentiate w.r.t. s

$$\begin{aligned} \int_0^L (-t) t e^{-st} dt &= -t^2 e^{-st} \Big|_0^L - \int_0^L \frac{d}{dt} (-t^2 e^{-st}) dt \\ &= -t^2 \frac{e^{-st}}{-s} \Big|_0^L - \int_0^L \frac{2t e^{-st}}{-s} dt \\ &= + \frac{2}{s^3} \end{aligned}$$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

Differentiate (n-1) times
w.r.t. s

$$L\{t^n\}$$

$$= \int_0^L t^n e^{-st} dt = \frac{n!}{s^{n+1}}$$

$$\begin{aligned} u dv &= uv - \int v du \\ &= -t^2 \frac{e^{-st}}{-s} \Big|_0^L - \int_0^L \frac{2t e^{-st}}{-s} dt \\ &= \int_0^L 2 \frac{e^{-st}}{s} t dt \\ &= \frac{2}{s} \left[\frac{e^{-st}}{(-s)} t \right]_0^L - \int_0^L \frac{e^{-st}}{-s} dt \end{aligned}$$

(V) $F(t) = e^{at}$

$$\begin{aligned} \therefore L\{e^{at}\} &= \int_0^L e^{at} e^{-st} dt = \int_0^L e^{-(s-a)t} dt \\ &= \frac{e^{-(s-a)t}}{-(s-a)} \Big|_0^\infty = \frac{1}{(s-a)} \end{aligned}$$

(VI) $f(t) = e^{-at}$

$$\begin{aligned} \therefore L\{e^{-at}\} &= \int_0^L e^{-st} e^{-at} dt \\ &= \int_0^L e^{-(s+a)t} dt = \frac{1}{s+a} \end{aligned}$$

Find the LT of t^n , $n > -1$

What is the LT of \sqrt{t}

Solution

$$L\{t^n\} = \int_0^\infty e^{-st} t^n dt$$

Let $x = st$ $dt = \frac{dx}{s}$, $t = \frac{x}{s}$

$$L\{t^n\} = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$$

$$= \int_0^\infty e^{-x} \frac{x^n}{s^{n+1}} dx = \int_0^\infty e^{-x} x^n dx \cdot \frac{1}{s^{n+1}}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n dx = \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^{(n+1)-1} dx$$

$$= \frac{1}{s^{n+1}} \Gamma(n+1) \quad \text{for } s > 0 \text{ and } (n+1) > 0$$

Let $L\{\sqrt{t}\} = L\{t^{1/2}\}$

$$L\{t^n\} = \frac{1}{s^{n+1}}$$

$$L\{t^{1/2}\} = \frac{\Gamma(1/2+1)}{s^{1/2+1}} = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{1/2 \sqrt{\pi}}{s^{3/2}} = \frac{1}{2} \frac{\sqrt{\pi}}{s^{5/2}}$$

$$\begin{aligned} & \overbrace{e^{-x} x^{n-1} dx} \\ & = \Gamma_n \end{aligned}$$

$$L\{t^{1/2}\} = \frac{1}{2s} \sqrt{\frac{\pi}{s}}$$

Find the L.T of -

(75)

- (i) $\sinh at$ (ii) $\cosh at$ (iii) $\sin at$
(iv) $\cos at$

Solution

$$\mathcal{L}\{F(t)\} = \int_0^{\infty} e^{-st} F(t) dt$$

(i) $F(t) = \sinh at$

$$\mathcal{L}\{\sinh at\} = \int_0^{\infty} e^{-st} \sinh at dt$$

$$= \int_0^{\infty} e^{-st} \frac{e^{at} - e^{-at}}{2} dt$$

$$= \frac{1}{2} \int_0^{\infty} [e^{-(s-a)t} - e^{-(s+a)t}] dt$$

$$= \frac{1}{2} \left[\frac{1}{s-a} - \frac{1}{s+a} \right] = \frac{1}{2} \frac{(s+a) - (s-a)}{(s-a)(s+a)}$$

$$= \frac{1}{2} \frac{(2a)}{s^2 - a^2} = \left(\frac{a}{s^2 - a^2} \right), \quad s > a$$

(i) $F(t) = \cosh at$

$$\mathcal{L}\{\cosh at\} = \int_0^{\infty} e^{-st} \cosh at dt$$

$$= \int_0^{\infty} e^{-st} \frac{e^{at} + e^{-at}}{2} dt$$

$$= \frac{1}{2} \int_0^{\infty} [e^{-(s-a)t} + e^{-(s+a)t}] dt$$

$$= \frac{1}{2} \left[\frac{1}{s-a} + \frac{1}{s+a} \right] = \frac{s}{s^2 - a^2}$$

$$(iii) F(s) = \text{final}$$

(76)

$$\mathcal{L}\{\text{final}\} = \int_0^{\infty} e^{-st} \text{final} dt$$

$$= \int_0^{\infty} e^{-st} \left(e^{iat} - e^{-iat} \right) dt$$

$$= \frac{1}{2i} \int_0^{\infty} \left[e^{-(s-i\alpha)t} - e^{-(s+i\alpha)t} \right] dt$$

$$= \frac{1}{2i} \left[\frac{1}{s-i\alpha} - \frac{1}{s+i\alpha} \right] = \frac{1}{2i} \left[\frac{\alpha i \alpha}{s^2 + \alpha^2} \right]$$

$$= \frac{\alpha}{s^2 + \alpha^2}$$

$$\text{IV } F(s) = \text{cosal}$$

$$\mathcal{L}\{\text{cosal}\} = \int_0^{\infty} e^{-st} \text{cosal} dt$$

$$= \int_0^{\infty} e^{-st} \left(e^{iat} + e^{-iat} \right) dt$$

$$= \frac{1}{2} \int_0^{\infty} \left[e^{-(s-i\alpha)t} + e^{-(s+i\alpha)t} \right] dt$$

$$= \frac{1}{2} \left[\frac{1}{s-i\alpha} + \frac{1}{s+i\alpha} \right] = \frac{s}{s^2 + \alpha^2}$$

Find the LT of

(i) $e^{at} \cos wt$ (ii) $e^{at} \sin wt$

$$(i) \mathcal{L}\{F(t)\} = \int_0^{\infty} e^{-st} F(t) dt$$

$$= \mathcal{L}\{e^{at} \cos wt\} = \int_0^{\infty} e^{-st} e^{at} \cos wt dt$$

$$= \int_0^{\infty} e^{-st} e^{at} \left(\frac{e^{i\omega t} + e^{-i\omega t}}{2} \right) dt \quad (77)$$

$$= \frac{1}{2} \int_0^{\infty} \left[e^{-(s-a-i\omega)t} + e^{-(s-a+i\omega)t} \right] dt$$

$$= \frac{1}{2} \left[\frac{1}{s-a-i\omega} + \frac{1}{s-a+i\omega} \right] = \frac{s-a}{(s-a)^2 + \omega^2}$$

(ii) $F(t) = e^{at} \sin \omega t$

$s > a$, a & ω real.

$$\mathcal{L} \{ e^{at} \sin \omega t \} = \int_0^{\infty} e^{-st} e^{at} \sin \omega t dt$$

$$= \int_0^{\infty} e^{-st} e^{at} \left(\frac{e^{i\omega t} - e^{-i\omega t}}{2i} \right) dt$$

$$= \frac{1}{2i} \int_0^{\infty} \left[e^{-(s-a-i\omega)t} - e^{-(s-a+i\omega)t} \right] dt$$

$$= \frac{1}{2i} \left[\frac{1}{s-a-i\omega} - \frac{1}{s-a+i\omega} \right]$$

$$= \frac{1}{2i} \left[\frac{2i\omega}{(s-a)^2 + \omega^2} \right] = \frac{\omega}{(s-a)^2 + \omega^2}$$

$s > a$,
 a & ω real.

Find the Laplace Transform

$$F(t) = \underline{e^{at} - 1}$$

Solve it

$$\mathcal{L} \{ F(t) \} = \mathcal{L} \left\{ e^{\underline{at}} - 1 \right\}$$

$$\begin{aligned}
 &= \frac{1}{a} L\{e^{at}\} - \frac{1}{a} L\{1\} \\
 &= \frac{1}{a} \int_0^{\infty} e^{-st} e^{at} dt - \frac{1}{a} \int_0^{\infty} e^{-st} 1 dt \\
 &= \frac{1}{a} \int_0^{\infty} e^{-(s-a)t} dt - \frac{1}{a} \int_0^{\infty} e^{-st} dt \\
 &= \frac{1}{a} \left(\frac{1}{s-a} \right) - \frac{1}{a} \frac{1}{s} = \frac{1}{a} \left[\frac{1}{s-a} - \frac{1}{s} \right] \\
 &= \frac{1}{a} \left[\frac{s - (s-a)}{(s-a)s} \right] = \frac{1}{s(s-a)}
 \end{aligned} \tag{78}$$

Find the $L\{f(t)\}$ if $f(t) = \sinh at$. final -
Solution

$$\begin{aligned}
 f(t) &= \sinh at = \frac{e^{at} - e^{-at}}{2i} = \frac{e^{iat} - e^{-iat}}{2i} \\
 &= \frac{1}{4i} \left[e^{ta(1+i)} + e^{-ta(1+i)} - e^{-ta(1-i)} - e^{ta(1-i)} \right] \\
 \text{Let } \beta &= a(1+i), \quad q = a(1-i) \\
 \therefore f(t) &= \frac{1}{4i} \left[e^{t\beta} + e^{-t\beta} - e^{-qt} - e^{qt} \right] \\
 L\{f(t)\} &= L\left\{ \frac{1}{4i} \left[e^{t\beta} + e^{-t\beta} - e^{-qt} - e^{qt} \right] \right\} \\
 &= \frac{1}{4i} \left[L\{e^{t\beta}\} + L\{e^{-t\beta}\} - L\{e^{-qt}\} \right. \\
 &\quad \left. - L\{e^{qt}\} \right]
 \end{aligned}$$

$$L\{e^{t\beta}\} = \int_0^{\infty} e^{-st} e^{t\beta} dt = \int_0^{\infty} e^{t(\beta-s)} dt = \frac{1}{s-\beta}$$

$$L\{e^{-t\beta}\} = \frac{1}{s+\beta}$$

$\therefore \left\{ \text{sink at final} \right\}$

(79)

$$= \frac{1}{4i} \left[\frac{1}{(s-p)} + \frac{1}{(s+p)} - \frac{1}{(s-q)} - \frac{1}{(s+r)} \right]$$

$$= \frac{1}{4i} \left\{ \frac{2s}{s^2-p^2} - \frac{2s}{(s^2-q^2)} \right\}$$

$$= \frac{1}{2i} \left\{ \frac{s}{(s^2-p^2)} - \frac{s}{(s^2-q^2)} \right\}$$

$$= \frac{1}{2i} \left[\frac{s(s^2-q^2) - s(s^2-p^2)}{(s^2-p^2)(s^2-q^2)} \right]$$

$$= \frac{1}{2i} \frac{[s^3s - s^2q^2 - s^3s + sp^2]}{(s^2-p^2)(s^2-q^2)}$$

$$= \frac{1}{2i} \left[\frac{-s(q^2-p^2)}{(s^2-p^2)(s^2-q^2)} \right]$$

$$= \frac{-i}{2} \left[\frac{s(p^2-q^2)}{(s^2-p^2)(s^2-q^2)} \right]$$

$$p^2 = (a(1+i))^2 = a^2 (1+i)^2 = a^2 [1+i^2 + 2i] \\ = a^2 [2i] = 2ia^2$$

$$q^2 = a^2 (1-i)^2 = a^2 (1^2 + i^2 - 2i) = -2ia^2$$

$$-i(p^2-q^2) = -i(2ia^2 + 2ia^2) = 4a^2$$

$$(s^2-p^2)(s^2-q^2) = (s^2-2ia^2)(s^2+2ia^2)$$

$$= s^4 + 4a^4$$

$$\therefore \left\{ \text{sink at final} \right\} = \frac{4q^2s}{2(s^4+4a^4)} = \frac{2q^2s}{s^4+4a^4}$$

Laplace transform of a derivative

(80)

Let $F(t)$ be a continuous differentiable

fn. Find $\frac{dF(t)}{dt}$ & its first derivative.

The Laplace transform of the derivative

$\frac{dF(t)}{dt}$ is given by,

$$L\left\{\frac{dF(t)}{dt}\right\} = sL\{F(t)\} - F(0)$$

$$= s \cdot f(s) - F(0)$$

where $F(0)$ is the value of $F(t)$ at $t = 0$

and $f(s) = L\{F(t)\}$.

Proof.

$$L\left\{\frac{dF(t)}{dt}\right\} = \int_0^\infty e^{-st} \frac{dF(t)}{dt} dt$$

Integrate by parts,

$$\begin{aligned} L\left\{\frac{dF(t)}{dt}\right\} &= \left[F(t) e^{-st} \right]_0^\infty + s \int_0^\infty e^{-st} F(t) dt \\ &= -F(0) + s \cdot L\{F(t)\} \\ &= s \cdot f(s) - F(0) \end{aligned}$$

$$\begin{aligned} \text{Hence } L\left\{\frac{d^2F(t)}{dt^2}\right\} &= s^2 L\{F(t)\} - s F(0) \\ &\quad - \left. \frac{dF}{dt} \right|_{t=0} \\ &= s^2 f(s) - s F(0) - \left. \frac{dF}{dt} \right|_{t=0} \end{aligned}$$

(81)

Initial value theorem

Corollary I If $\mathcal{L}\{F(t)\} = f(s)$, then $\lim_{t \rightarrow 0} F(t) = \lim_{s \rightarrow \infty} s f(s)$

Proof

$$\mathcal{L}\left\{\frac{dF}{dt}\right\} = \mathcal{L}\{f'(t)\} - F(0)$$

$$\text{or } \int_0^{\infty} e^{-st} \left(\frac{dF}{dt}\right) dt = s f(s) - F(0)$$

Taking limit $s \rightarrow \infty$,

$$\lim_{s \rightarrow \infty} \int_0^{\infty} e^{-st} \left(\frac{dF}{dt}\right) dt = \lim_{s \rightarrow \infty} [s f(s) - F(0)]$$

LHS $\rightarrow 0$, as $s \rightarrow \infty$,

$$\therefore \lim_{s \rightarrow \infty} [s f(s) - F(0)] = 0$$

$$\lim_{s \rightarrow \infty} s f(s) - F(0) = 0$$

$$\lim_{s \rightarrow \infty} s f(s) = F(0)$$

 $s \rightarrow \infty$

But $F(0)$ obtained from $F(t)$ as $t \rightarrow 0$
 $\therefore F(0) = \lim_{t \rightarrow 0} F(t)$

$$\therefore \lim_{s \rightarrow \infty} s f(s) = F(0) = \lim_{s \rightarrow 0} F(t)$$

Corollary IIFinal value theorem

$$\text{If } \mathcal{L}\{f(t)\} = F(s), \quad (8a)$$

$$\text{then } \lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} s F(s)$$

Proof

$$\mathcal{L}\left\{\frac{dF}{dt}\right\} = \int_0^{\infty} e^{-st} \frac{dF}{dt} dt$$

$$= s F(s) - F(0)$$

$$\lim_{s \rightarrow 0} \int_0^{\infty} e^{-st} \left(\frac{dF}{dt}\right) dt = \lim_{s \rightarrow 0} [s F(s) - F(0)]$$

If $f(s) \rightarrow 0$, then LHS becomes

$$\int_0^{\infty} \frac{dF}{dt} (dt) = [F(t)]_0^{\infty} = \lim_{t \rightarrow \infty} F(t) - F(0)$$

$$\therefore \cancel{\int_0^{\infty} F(t) dt} = \lim_{t \rightarrow \infty} F(t) - F(0)$$

$$\therefore \lim_{t \rightarrow \infty} F(t) - F(0) = \lim_{s \rightarrow 0} s f(s) - F(0)$$

$$\lim_{t \rightarrow \infty} F(t) = \lim_{s \rightarrow 0} s f(s)$$

Laplace transform of integral.

$$\text{If } f(s) = \mathcal{L}\{F(t)\}$$

Then, $\mathcal{L}\left[\int_0^s F(t) dt\right] = \frac{f(s)}{s}$

Proof we have
 Given $f(s) = L\{F(t)\}$

(83)

Let $G(t) = \int_0^t F(\tau) d\tau$

$$G(0) = \int_0^\infty F(\tau) d\tau = 0$$

$$G'(t) = \frac{d}{dt} \int_0^t F(\tau) d\tau = F(t)$$

We know,

$$L\{G'(t)\} = s L\{G(t)\} - G(0)$$

$$L\{F(t)\} = s L\{G(t)\} - 0$$

$$L\{F(t)\} = s L\{G(t)\}$$

$$\therefore L\{F(t)\} = L\{G(t)\}$$

$$\begin{aligned} \therefore L\left\{\int_0^t F(\tau) d\tau\right\} &= \frac{1}{s} L\{F(t)\} \\ &= \frac{1}{s} f(s) = \frac{f(s)}{s} \end{aligned}$$

Inverse Laplace transform.

(84)

Fourier Merlin theorem.

If the LT of a fn. $f(s)$ is $F(t)$, i.e., $\mathcal{L}\{F(t)\} = f(s)$ then $F(t)$ is called the inverse LT of $f(s)$.

$$F(t) = \mathcal{L}^{-1}\{f(s)\}$$

\mathcal{L}^{-1} is the inverse LT operator.

If $F(t)$ is piecewise differentiable fn, then its Fourier transform integral is

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iut} \left[\int_{-\infty}^{+\infty} F(x) e^{-ixu} dx \right] du$$

Assume that $F(t)$ has the property that $F(t) = 0$ for $t < 0$

$$\therefore F(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iut} \left[\int_0^{\infty} F(x) e^{-ixu} dx \right] du$$

Consider a fn. $G(t)$ such that

$$G(t) = e^{-ct} F(t)$$

where c is a positive constant.

The Fourier integral of $G(t)$ is

$$G(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iut} \left[\int_0^{\infty} G(x) e^{-ixu} dx \right] du$$

$$\begin{aligned} G(t) &= e^{-ct} F(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iut} \left[\int_0^{\infty} e^{-cx} F(x) e^{-ixu} dx \right] du \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iut} \left[\int_0^{\infty} F(x) e^{-(c+iu)x} dx \right] du \end{aligned}$$

$$\text{Left } s = c + iu, \quad u = \frac{s-c}{i} = \frac{s}{i} - \frac{c}{i}$$

$$iu = s - c, \quad du = \frac{ds}{i}$$

$$G(t) = e^{-ct} F(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{(s-c)t} \left[\int_0^{\infty} F(x) e^{-sx} dx \right] ds$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \left[\int_0^{\infty} F(x) e^{-sx} dx \right] ds$$

(or)

~~$$F(t) = \frac{G(t)}{e^{-ct}}$$~~

$$\frac{G(t)}{e^{-ct}} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \left[\int_0^{\infty} F(x) e^{-sx} dx \right] ds$$

We have $f(s) = \int_0^{\infty} F(x) e^{-sx} dx$

$$= \int_0^{\infty} F(t) e^{-st} dt$$

$$F(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} f(s) ds$$

\rightarrow Bromwich integral
 \rightarrow Fourier Mellin integral

Find the inverse LT of (i)

~~$s^2 + a^2$~~

Solution

~~$$f(s) = \frac{1}{s^2 + a^2}$$~~

86

Properties of inverse Laplace transform

① Linearity property

If $f_1(s)$ and $f_2(s)$ are $L.T$ of $F_1(t)$ and $F_2(t)$, respectively, then,

$$\begin{aligned} L^{-1} \{ c_1 f_1(s) + c_2 f_2(s) \} \\ = c_1 L^{-1} \{ f_1(s) \} + c_2 L^{-1} \{ f_2(s) \}. \end{aligned}$$

where c_1 & c_2 are constants.

Proof

By linearity property of Laplace transforms, we have

$$\begin{aligned} L \{ c_1 F_1(t) + c_2 F_2(t) \} &= c_1 L \{ F_1(t) \} + \\ &\quad c_2 L \{ F_2(t) \} \\ &= c_1 f_1(s) + c_2 f_2(s) \end{aligned}$$

Taking inverse $L.T$,

$$\begin{aligned} c_1 F_1(t) + c_2 F_2(t) &= L^{-1} \{ c_1 f_1(s) + c_2 f_2(s) \} \\ &= c_1 L^{-1} \{ f_1(s) \} + c_2 L^{-1} \{ f_2(s) \} \\ &= L^{-1} \{ c_1 f_1(s) + c_2 f_2(s) \} \end{aligned}$$

② Change of scale property

If $f(s)$ is the LT of a fn $f(t)$,
 Then $\mathcal{L}^{-1}\{f(as)\} = \frac{F(\frac{t}{a})}{a}, a > 0$

Proof

we have $f(s) = \int_0^{\infty} e^{-st} F(t) dt$

$$\begin{aligned} f(as) &= \int_0^{\infty} e^{-ast} F(t) dt \\ &= \int_0^{\infty} e^{-sx} \text{Let } at = x, t = \frac{x}{a}, dt = \frac{dx}{a} \\ &\quad F\left(\frac{x}{a}\right) \frac{dx}{a} \\ &= \frac{1}{a} \int_0^{\infty} e^{-sx} F\left(\frac{x}{a}\right) dx \\ &= \frac{1}{a} \int_0^{\infty} e^{-st} F\left(\frac{t}{a}\right) dt \\ &= \frac{1}{a} \mathcal{L}\left\{F\left(\frac{t}{a}\right)\right\} \end{aligned}$$

③ First translation' or shifting property-

If $f(s)$ is the LT of $F(t)$ Then,
 $\mathcal{L}^{-1}\{f(s-a)\} = e^{at} F(t)$

Proof-

$$f(s) = \int_0^{\infty} e^{-st} F(t) dt$$

$$\begin{aligned} f(s-a) &= \int_0^{\infty} e^{-(s-a)t} F(t) dt \\ &= \int_0^{\infty} e^{-st} \left\{ e^{at} F(t) \right\} dt = \end{aligned}$$

$$= L \left\{ e^{at} F(t) \right\} \quad (88)$$

$$\therefore L^{-1} \left\{ f(s-a) \right\} = e^{at} F(t)$$

116)

$$L^{-1} \left\{ f(s+a) \right\} = e^{-at} F(t)$$

④ Second shifting property

If $L^{-1} \left\{ f(s) \right\} = F(t)$ then

$$L^{-1} \left\{ e^{-as} f(s) \right\} = G(t) = \begin{cases} 0; t < a \\ F(t-a); t > a \end{cases}$$

Proof

$$L \left\{ G(t) \right\} = \int_0^{\infty} e^{-st} G(t) dt$$

$$= \int_0^a e^{-st} G(t) dt + \int_a^{\infty} e^{-st} F(t-a) dt$$

$$= 0 + \int_a^{\infty} e^{-st} F(t-a) dt$$

$$= \cancel{\int_a^{\infty} e^{-st} ds} \quad \text{Let } t-a=y \\ \quad \quad \quad t = y+a \\ \quad \quad \quad dt = dy$$

$$= \int_a^{\infty} e^{-s(y+a)} F(y) dy$$

$$= \int_0^{\infty} e^{-sy} e^{-sa} F(y) dy \quad (89)$$

$$\begin{aligned} L\{g(t)\} &= e^{-sa} \int_0^{\infty} e^{-sy} F(y) dy = e^{-sa} L\{f(t)\} \\ L\{g(t)\} &= e^{-sa} f(s) \end{aligned}$$

Take inverse transform,

$$c(t) = L^{-1}\left\{e^{-sa} f(s)\right\}$$

Find the inverse L^{-1} of,

$$(i) \frac{1}{\sqrt{2s+5}} \quad (ii) \frac{e^{-\frac{1}{2}s}}{s}$$

~~(i) $L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = t^n$~~

~~$L\{t^n\} = \Gamma(n+1) \left(\frac{1}{s^{n+1}}\right)$~~

$$L^{-1}\left\{L\{t^n\}\right\} = t^n = L^{-1}\left\{\frac{\Gamma(n+1)}{s^{n+1}}\right\}$$

$$\frac{t^n}{\Gamma(n+1)} = L^{-1}\left\{\frac{1}{s^{n+1}}\right\}$$

$$\therefore L^{-1}\left\{\frac{1}{2s+5}\right\} = L^{-1}\left\{\frac{1}{\sqrt{2s+5}}\right\}$$

$$= L^{-1}\left\{\frac{1}{\cancel{s}\sqrt{(2s+\cancel{5})/2}}\right\}$$

$$= L^{-1}\left\{\frac{1}{\sqrt{(s+5/2)/2}}\right\} = L^{-1}\left\{\frac{1}{\sqrt{2}(s+5/2)^{1/2}}\right\}$$

$$= L^{-1} \left\{ \frac{1}{s_2} \frac{1}{(s+s_2)^{\frac{1}{2}}} \right\} \quad (90)$$

$$= \frac{1}{s_2} L^{-1} \left\{ \frac{1}{(s+s_2)^{\frac{1}{2}}} \right\} = \frac{1}{s_2} e^{-s_2 t} L^{-1} \left\{ \frac{1}{s^{\frac{1}{2}}} \right\}$$

$$= \frac{1}{s_2} e^{-s_2 t} \frac{t^{-\frac{1}{2}}}{\Gamma^{\frac{1}{2}}} \quad (n = -\frac{1}{2})$$

$$= \frac{1}{s_2} e^{-s_2 t} \frac{t^{-\frac{1}{2}}}{\sqrt{\pi}} = \frac{1}{s_2 \sqrt{\pi}} e^{-s_2 t} t^{-\frac{1}{2}}$$

$$(ii) \quad \frac{e^{-\frac{1}{2}s}}{s} = \frac{1}{s} \left[1 - \frac{1}{s} + \frac{1}{s^2 2!} - \frac{1}{s^3 3!} + \dots \right]$$

$$= \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3 2!} - \frac{1}{s^4 3!} + \dots$$

Take inverse transform both sides

$$\left(\text{we know } L^{-1} \left\{ \frac{n!}{s^{n+1}} \right\} = t^n \right) \quad \left[L^{-1} \left\{ \frac{1}{s^{n+1}} \right\} = \frac{t^n}{n!} \right]$$

$$L^{-1} \left\{ \frac{e^{-\frac{1}{2}s}}{s} \right\} = L^{-1} \left\{ \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3 2!} - \frac{1}{s^4 3!} + \dots \right\}$$

$$\Rightarrow \qquad = 1 - t + \frac{t^2}{2! 2!} - \frac{t^3}{3! 3!} + \dots$$

$$= 1 - t + \frac{t^2}{(2!)^2} - \frac{t^3}{(3!)^2}$$

$$= 1 + \frac{(2\sqrt{t})^2}{2^2} + \frac{(2\sqrt{t})^4}{2^2 4^2} - \frac{(2\sqrt{t})^6}{2^2 4^2 6^2}$$

$$= J_0(2\sqrt{t})$$

Find the inverse L.T of

(9B)

$$\frac{e^{-\pi s}}{s^2 + 1}$$

Signal $\rightarrow \frac{a}{s^2 + a^2}$

$$\stackrel{\text{def}}{=} \mathcal{L} \left\{ \sin t \right\} = \int_0^\infty e^{-st} \sin t dt = \frac{1}{s^2 + 1}$$

$$\therefore \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 1} \right\} = \sin t = F(t)$$

By shifting property -

$$\mathcal{L}^{-1} \left\{ \frac{e^{-\pi s}}{s^2 + 1} \right\} = \sin(t - \pi) U(t - \pi)$$

where $U(t - \pi) = \begin{cases} 1 & \text{for } t > \pi \\ 0 & \text{for } t < \pi \end{cases}$

Solution of differential equation

Consider

$$a_0 \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_2 x = F(t)$$

where a_0, a_1 & a_2 are coefficients (constants).

Take L.T both sides,

$$a_0 \mathcal{L} \left\{ \frac{d^2 x}{dt^2} \right\} + a_1 \mathcal{L} \left\{ \frac{dx}{dt} \right\} + a_2 \mathcal{L} \left\{ x \right\} = \mathcal{L} \{ F(t) \}$$

But from property of L.T. derivative,

$$\mathcal{L} \left\{ \frac{dx}{dt} \right\} = s \mathcal{L} \{ x \} - [x]_{t=0}$$

$$\mathcal{L} \left\{ \frac{d^2 x}{dt^2} \right\} = s^2 \mathcal{L} \{ x \} - s \{ x \}_{t=0} - \left\{ \frac{dx}{dt} \right\}_{t=0}$$

$$\mathcal{L}\{f(t)\} = f(s)$$

(9a)

$$\therefore a_0 \left[s^2 \mathcal{L}\{x\} - x|_{t=0} - \left(\frac{dx}{dt} \right)_{t=0} \right] +$$

$$a_1 \left[s \mathcal{L}\{x\} - x|_{t=0} \right] +$$

$$a_2 \mathcal{L}\{x\} = f(s)$$

$$(a_0 s^2 + a_1 s + a_2) \mathcal{L}\{x\} = f(s) + (a_0 s + a_1) x|_{t=0} + a_0 \left(\frac{dx}{dt} \right)_{t=0}$$

$$\mathcal{L}\{x\} = \underbrace{f(s) + (a_0 s + a_1) x|_{t=0} + a_0 \left(\frac{dx}{dt} \right)_{t=0}}_{a_0 s^2 + a_1 s + a_2}$$

The RHS is in the form

$\frac{G(s)}{H(s)}$: the solution $x(t)$

$$= \mathcal{L}^{-1} \left\{ \frac{G(s)}{H(s)} \right\}$$

Using LT, solve the differential equation

$$y'' + qy = 0. \quad \text{Solving initial conditions } y(0) = 0, \quad y'(0) = 2.$$

$$\text{Given, } \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + q} \right\} = \sin \sqrt{q} t$$

$$y'' + qy = 0$$

Take LT both sides

$$\mathcal{L}\{y''\} + q \mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$s^2 \mathcal{L}\{y\} - s\{y\}_{t=0} - \left[\frac{dy}{dt} \right]_{t=0} + q \mathcal{L}\{y\} = \{0\} \quad (93)$$

$$\begin{aligned} (s^2 + q) \mathcal{L}\{y\} &= \{0\} + s\{y\}_{t=0} + \left[\frac{dy}{dt} \right]_{t=0} \\ &= \{0\} + s y(0) + y'(0) \\ &= 0 + s \cdot 0 + 2 = 2 \end{aligned}$$

$$\mathcal{L}\{y\} = \frac{2}{(s^2 + q)} = \frac{2}{3} \frac{3}{(s^2 + q)}$$

~~Take inverse L~~

$$\begin{aligned} \mathcal{L}^{-1} \mathcal{L}\{y\} &= y = \mathcal{L}^{-1} \left\{ \frac{2}{3} \frac{3}{(s^2 + q)} \right\} \\ &= \frac{2}{3} \mathcal{L}^{-1} \left\{ \frac{3}{s^2 + q} \right\} = \frac{2}{3} \sin 3t \end{aligned}$$

Solve the differential equation

$$\frac{d^2x}{dt^2} - 2 \frac{dx}{dt} + 2x = 0; \quad x_0 = x_1 = 1$$

Given

$$\text{Given } D^2x - 2 Dx + 2x = 0$$

$$\text{or } x'' - 2x' + 2x = 0$$

Take L.T both sides

$$\mathcal{L}\{x''\} - 2 \mathcal{L}\{x'\} + 2 \mathcal{L}\{x\} = \{0\}$$

$$\begin{aligned} s^2 \mathcal{L}\{x\} - s\{x\}_{t=0} - [x']_{t=0} - 2 \left[s \mathcal{L}\{x\} + [x']_{t=0} \right] \\ + 2 \mathcal{L}\{x\} = \{0\} \end{aligned}$$

(94)

$$(s^2 - 2s + 2) L\{x\}$$

$$= L\{b\} + (s-2) \{x\}_{t=0} + \{x'\}_{t=0}$$

$$= 0 + (s-2) + 1 = s-1$$

$$L\{x\} = \frac{s-1}{s^2 - 2s + 2}$$

Take inverse transform

$$\leftarrow x(t) = L^{-1} \left\{ \frac{s-1}{s^2 - 2s + 2} \right\} = e^t \cos t$$